

PARABOLIC SHEAVES ON SURFACES AND AFFINE LIE ALGEBRA $\widehat{\mathfrak{gl}}_n$

MICHAEL FINKELBERG AND ALEXANDER KUZNETSOV

1. INTRODUCTION

The purpose of this paper is to give an example of geometric construction (via Hecke correspondences) of certain representations of the affine Lie algebra $\widehat{\mathfrak{gl}}_n$. The construction is similar to the one of [3] for the Lie algebra \mathfrak{sl}_n .

1.1. The case of \mathfrak{sl}_n . Recall the setup of [3]. Let V be an n -dimensional complex vector space, and let $\alpha = (a_1, \dots, a_{n-1})$ be an $(n-1)$ -tuple of nonnegative integers. We consider α as a linear combination $\alpha = \sum a_i i \in \mathbb{N}[I]$ of simple coroots $i \in I$ of the Lie algebra \mathfrak{sl}_n via identification $I = \{1, 2, \dots, n-1\}$. Let \mathcal{Q}_α^L be the space of degree α quasiflags, that is the space of all flags $0 \subset E_1 \subset \dots \subset E_{n-1} \subset V \otimes \mathcal{O}_C$ of coherent subsheaves in the trivial vector bundle $V \otimes \mathcal{O}_C$ over the curve $C = \mathbb{P}^1$ such that

$$\text{rank}(E_p) = p, \quad \deg(E_p) = -a_p.$$

The space \mathcal{Q}_α^L is a smooth compactification of the space \mathcal{Q}_α of degree α maps from the curve C to the flag variety \mathbf{X} of the Lie group $SL(V)$ (see [8]). The subspace $\mathcal{Q}_\alpha \subset \mathcal{Q}_\alpha^L$ is formed by all degree α flags of vector subbundles $0 \subset E_1 \subset \dots \subset E_{n-1} \subset V \otimes \mathcal{O}_C$.

For $i \in I$ let $\mathfrak{E}_\alpha^i \subset \mathcal{Q}_\alpha^L \times \mathcal{Q}_{\alpha+i}^L$ denote the closed subspace consisting of pairs of flags (E_\bullet, E'_\bullet) such that $E'_\bullet \subset E_\bullet$. The subspace $\mathfrak{E}_\alpha^i \subset \mathcal{Q}_\alpha^L \times \mathcal{Q}_{\alpha+i}^L$ considered as a correspondence defines a pair of operators

$$e_i : H^\bullet(\mathcal{Q}_\alpha^L, \mathbb{Q}) \rightarrow H^{\bullet+2}(\mathcal{Q}_{\alpha+i}^L, \mathbb{Q}) \quad \text{and} \quad f_i : H^\bullet(\mathcal{Q}_{\alpha+i}^L, \mathbb{Q}) \rightarrow H^{\bullet-2}(\mathcal{Q}_\alpha^L, \mathbb{Q})$$

Finally, let $h_i : H^\bullet(\mathcal{Q}_\alpha^L, \mathbb{Q}) \rightarrow H^\bullet(\mathcal{Q}_\alpha^L, \mathbb{Q})$ be the scalar multiplication by $2 + 2a_i - a_{i-1} - a_{i+1}$.

Theorem ([3]). *The operators e_i , f_i , h_i provide the vector space*

$$\mathbf{H} := \bigoplus_{\alpha \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}_\alpha^L, \mathbb{Q})$$

with a structure of \mathfrak{sl}_n -module.

1.2. Verma submodules. Let $\mathcal{E}_\bullet \in \mathcal{Q}_{\alpha_0} \subset \mathcal{Q}_{\alpha_0}^L$ be a flag of vector subbundles of degree α_0 . Let $K_\alpha(\mathcal{E}_\bullet) \subset \mathcal{Q}_{\alpha+\alpha_0}^L$ denote the closed subspace formed by all quasiflags E_\bullet such that $E_\bullet \subset \mathcal{E}_\bullet$. It is equidimensional of dimension $|\alpha| = a_1 + \dots + a_{n-1}$. Consider the vector subspace

$$\mathbf{M}(\mathcal{E}_\bullet) := \bigoplus_{\alpha \in \mathbb{N}[I]} H^0(K_\alpha(\mathcal{E}_\bullet), \mathbb{Q}) \subset \mathbf{H}$$

spanned by the fundamental cycles of the irreducible components of $K_\alpha(\mathcal{E}_\bullet)$. It is a \mathfrak{sl}_n -submodule of \mathbf{H} , isomorphic to the Verma module with the lowest weight $\alpha_0 + 2\rho$.

Let us describe the \mathfrak{sl}_n -module structure on $\mathbf{M}(\mathcal{E}_\bullet)$ in the intrinsic terms of the spaces $K_\alpha(\mathcal{E}_\bullet)$. We fix the flag \mathcal{E}_\bullet and we will write K_α instead of $K_\alpha(\mathcal{E}_\bullet)$ for brevity. We have to compute the matrices of the operators e_i and f_i in the bases of fundamental cycles of top dimensional irreducible components of $K_\alpha, K_{\alpha+i}$. If one views α as an element of the coroot lattice of \mathfrak{sl}_n , then $|\alpha|$ -dimensional irreducible components of the space K_α are in one-to-one correspondence with Kostant partitions of α (see [7]). Given a Kostant partition $A \in \mathfrak{K}(\alpha)$ we denote by $\overline{K}_A \subset K_\alpha$ the corresponding irreducible component and by $v_A \in H^0(K_\alpha, \mathbb{Q})$ the fundamental class of \overline{K}_A .

Let $A \in \mathfrak{K}(\alpha)$, $A' \in \mathfrak{K}(\alpha + i)$ be Kostant partitions. In order to compute the matrix coefficient $\varepsilon_i(A, A')$ of the operator e_i we should describe the intersection

$$\mathfrak{E}_A^{A'} := \mathfrak{E}_\alpha^i \cap p^{-1}(\overline{K}_A) \cap q^{-1}(\overline{K}_{A'}),$$

where $p : \mathfrak{E}_\alpha^i \rightarrow \mathcal{Q}_\alpha^L$ and $q : \mathfrak{E}_\alpha^i \rightarrow \mathcal{Q}_{\alpha+i}^L$ are the projections. More precisely, we need to know all irreducible components of $\mathfrak{E}_A^{A'}$ which are dominant over $\overline{K}_{A'}$. One can check that all these components have the expected dimension and the intersection $\mathfrak{E}_\alpha^i \cap p^{-1}(\overline{K}_A)$ is transversal along them. Hence the matrix coefficient $\varepsilon_i(A, A')$ is equal to the sum of degrees of these components over $\overline{K}_{A'}$.

Similarly, in order to compute the matrix coefficient $\phi_i(A', A)$ of the operator f_i we need to know all irreducible components of $\mathfrak{E}_A^{A'}$ which are dominant over \overline{K}_A . However, in contrast to the case of e_i , the situation here is rather complicated. Namely, in some cases these irreducible components have the expected dimension and then the matrix coefficient is equal to the degree of these components over \overline{K}_A . But in some cases the dimension of these irreducible components exceeds the expected dimension by 1 (the excess intersection). In these cases we also need to describe the excess intersection line bundle on these components. Then the matrix coefficient is equal to the sum of the degrees of the restrictions of this line bundle to a generic fiber of these components over \overline{K}_A .

1.3. The case of $\widehat{\mathfrak{gl}}_n$: a wishful thinking. Let $C \subset S$ be a smooth compact curve of genus $g = g(C)$ in a smooth compact surface S . Let us fix a sequence $\gamma = (c_i)$, $i \in \mathbb{Z}$, of cohomology classes in $H^2(S, \mathbb{Z})$, such that $c_{i+1} = c_i + [C]$, and a sequence $\alpha = (a_i)$, $i \in \mathbb{Z}$, of cohomology classes in $H^4(S, \mathbb{Z}) = \mathbb{Z}$ such that $a_{i+n} = a_i + c_i \cdot [C] + n[C]^2/2$. Recall that a parabolic sheaf of rank n on (S, C) is an infinite flag of torsion free rank n coherent sheaves $\dots \subset E_{-1} \subset E_0 \subset E_1 \subset \dots$ such that $E_{i+n} = E_i(C)$. Let $\mathcal{Y}(\gamma, \alpha)$ be the moduli space of μ -stable rank n parabolic sheaves such that $ch_1(E_i) = c_i$, $ch_2(E_i) = a_i$ (see [12]). For an n -periodic sequence of nonnegative integers $\beta = (b_i)$, let $\mathfrak{E}^\beta(\gamma, \alpha) \subset \mathcal{Y}(\gamma, \alpha) \times \mathcal{Y}(\gamma, \alpha + \beta)$ be the closed subspace formed by all pairs of parabolic sheaves (E_\bullet, E'_\bullet) such that $E'_\bullet \subset E_\bullet$, and $E_\bullet|_{S-C} = E'_\bullet|_{S-C}$.

We expect that $\mathfrak{E}^\beta(\gamma, \alpha)$ is equidimensional of dimension $\dim \mathcal{Y}(\gamma, \alpha) + |\beta|$ (where $|\beta| = \sum_{i=0}^{n-1} b_i$), and its irreducible components are parametrized by the set $\mathfrak{K}(\beta)$ of isomorphism classes of β -dimensional nilpotent representations of the cyclic quiver with n vertices (affine quiver of type $\widetilde{\mathbf{A}}_{n-1}$). For an isomorphism class $\kappa \in \mathfrak{K}(\beta)$ let us denote the corresponding irreducible component of $\mathfrak{E}^\beta(\gamma, \alpha)$ by $\mathfrak{E}_\kappa^\beta(\gamma, \alpha)$. The set of the above isomorphism classes over all $\beta \in \mathbb{N}[\mathbb{Z}/n\mathbb{Z}]$ forms a basis of the *Hall*

algebra \mathbf{H}_n of the category of nilpotent representations of the cyclic quiver $\tilde{\mathbf{A}}_{n-1}$. An irreducible component $\mathfrak{E}_\kappa^\beta(\gamma, \alpha)$ viewed as a correspondence between $\mathcal{Y}(\gamma, \alpha)$ and $\mathcal{Y}(\gamma, \alpha + \beta)$ defines the map $H^\bullet(\mathcal{Y}(\gamma, \alpha), \mathbb{Q}) \rightarrow H^\bullet(\mathcal{Y}(\gamma, \alpha + \beta), \mathbb{Q})$, and we expect that this way one obtains the action of \mathbf{H}_n on $\bigoplus_\alpha H^\bullet(\mathcal{Y}(\gamma, \alpha), \mathbb{Q})$.

According to [11], \mathbf{H}_n is isomorphic to the positive part of the enveloping algebra $U(\widehat{\mathfrak{gl}}_n)$. We expect that the above correspondences transposed define the action of the negative part of $U(\widehat{\mathfrak{gl}}_n)$, and together they generate the action of $U(\widehat{\mathfrak{gl}}_n)$.

1.4. Back to Verma. At the moment we are unable to carry out the above program. We have to restrict ourselves to the affine analog of 1.2. Namely, we fix a parabolic sheaf consisting of *locally free sheaves* $\dots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots$ For an n -periodic sequence of nonnegative integers $\alpha = (a_i)$ we define $K_\alpha = K_\alpha(\mathcal{E}_\bullet)$ as the space formed by all parabolic sheaves E_\bullet such that $E_\bullet \subset \mathcal{E}_\bullet$, and \mathcal{E}_i/E_i is concentrated on $C \subset S$ and has (finite) length a_i for any $i \in \mathbb{Z}$. We prove that K_α is equidimensional of dimension $|\alpha|$, and its irreducible components are naturally parametrized by the set $\mathfrak{K}(\alpha)$ of isomorphism classes of nilpotent representations of dimension α of the cyclic quiver $\tilde{\mathbf{A}}_{n-1}$. The fundamental classes of these components form a basis of $\mathbf{M} = \bigoplus_\alpha H^0(K_\alpha, \mathbb{Q})$. For $\kappa \in \mathfrak{K}(\alpha)$ we denote the corresponding basis element by v_κ .

For $i \in \mathbb{Z}/n\mathbb{Z}$ let $\alpha + i$ denote the sequence (a'_j) such that $a'_j = a_j + \delta_{ij}$. Let $\mathfrak{E}_\alpha^i \subset K_\alpha \times K_{\alpha+i}$ be the correspondence formed by all the pairs (E_\bullet, E'_\bullet) such that $E'_\bullet \subset E_\bullet$. It defines the maps

$$e_i : H^0(K_\alpha, \mathbb{Q}) \rightarrow H^0(K_{\alpha+i}, \mathbb{Q}), \quad \text{and} \quad f_i : H^0(K_{\alpha+i}, \mathbb{Q}) \rightarrow H^0(K_\alpha, \mathbb{Q}),$$

the matrix coefficients in the basis v_κ are defined similarly to 1.2. We prove that $e_i, f_i, i \in \mathbb{Z}/n\mathbb{Z}$, generate the action of $\widehat{\mathfrak{sl}}_n$ on \mathbf{M} .

The (restricted) dual space \mathbf{M}^* may be identified with the polynomial algebra $\mathbb{Q}[x_\theta]$ on infinitely many generators parametrized by the indecomposable nilpotent representations of $\tilde{\mathbf{A}}_{n-1}$. The dual action of (Chevalley generators of) $\widehat{\mathfrak{sl}}_n$ on \mathbf{M}^* is realized by the explicit first order differential operators in the coordinates x_θ .

On the other hand, $e_i \in \widehat{\mathfrak{sl}}_n$, $i \in \mathbb{Z}/n\mathbb{Z}$, generate the positive part of the universal enveloping algebra $U^+(\widehat{\mathfrak{sl}}_n)$ which is naturally embedded into \mathbf{H}_n (as the subalgebra generated by the isomorphism classes of indecomposable nilpotent representations). We write down explicit formulae for the action of $U(\widehat{\mathfrak{gl}}_n) \supset U^+(\widehat{\mathfrak{gl}}_n) = \mathbf{H}_n \supset U^+(\widehat{\mathfrak{sl}}_n)$ by differential operators in the coordinates x_θ . At the moment we cannot prove the geometric meaning behind these formulae (see though the Remark 7.3.7). Let us only mention that the central charge of \mathbf{M} equals $(2 - 2g(C))n + [C]^2$ (recall that $[C]^2$ equals the degree of the normal bundle $\mathcal{N}_{C/S}$).

1.5. Let us say a few words about the structure of the paper. In §2 we recall the necessary information about the nilpotent representations of cyclic quivers in various categories. In §3 we study the space K_α together with its projection to a configuration space of C . We prove that all the fibers of this projection admit a cell decomposition, and compute dimensions of all the cells. In §4 we study the correspondence $\mathfrak{E}_\alpha^i \subset K_\alpha \times K_{\alpha+i}$, and describe its irreducible components dominant over the topdimensional components of K_α and $K_{\alpha+i}$. It appears that for every component of \mathfrak{E}_α^i its projection to $K_{\alpha+i}$ is semismall. In §5 we define geometrically the matrix coefficients of the operators $e_i, f_i, h_i \in \widehat{\mathfrak{sl}}_n$ in the basis

of topdimensional components of K_α , $\alpha \in \mathbb{N}[\mathbb{Z}/n\mathbb{Z}]$, and compute them explicitly. In §6 we realize the $\widehat{\mathfrak{sl}}_n$ -module \mathbf{N} dual to \mathbf{M} of §5 in the polynomial functions on an infinite-dimensional affine space. The action of $\widehat{\mathfrak{sl}}_n$ on \mathbf{N} is given by explicit differential operators. This realization is similar to Kostant's construction of a dual Verma module over a semisimple Lie algebra in the sections of a line bundle over the big Schubert cell. Finally, in §7 we write down explicit differential operators extending the $\widehat{\mathfrak{sl}}_n$ -action on \mathbf{N} to $\widehat{\mathfrak{gl}}_n$. It is likely that the resulting $\widehat{\mathfrak{gl}}_n$ -module is a contragredient Verma module.

1.6. The work [3] might be viewed as a globalization of the geometric construction of $U(\mathfrak{sl}_n)$ discovered in [1], [4] (from a nilpotent neighbourhood of a point $x \in C$ to the global curve C). Similarly, the present work may be viewed as a globalization of the geometric construction of $U(\widehat{\mathfrak{sl}}_n)$ discovered in [5], [9]. In another direction, [3] was generalized in [2] to arbitrary simple finite dimensional algebras. It would be extremely important to generalize the present work along these lines.

Quite naturally, in this paper $n \geq 2$. But in fact, many considerations below make sense for $n = 1$ when the moduli space of parabolic sheaves becomes just the punctual Hilbert scheme of S . We were strongly influenced by the beautiful book [10], especially chapters 7–9 (see also [6]). In another direction, we were strongly motivated by the suggestion of V. Ginzburg back in 1997 to study the Drinfeld compactification of the space of maps from \mathbb{P}^1 to an affine Grassmannian. We learned from him about the parallel between Laumon and Drinfeld compactifications on the one hand, and Gieseker and Uhlenbeck compactifications, on the other.

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2. NOTATIONS

2.1. **Nilpotent representations of the cyclic quiver.** Let $\widetilde{\mathbf{A}}_{n-1}$ denote the cyclic quiver with n vertices and \mathbf{A}_∞ denote the infinite linear quiver. A representation M of the quiver \mathbf{A}_∞ is a \mathbb{Z} -graded vector space with an operator A of degree 1. A representation of the quiver $\widetilde{\mathbf{A}}_{n-1}$ is the same as a n -periodic representation of the quiver \mathbf{A}_∞ , that is a representation M with an isomorphism $M[n] \cong M$, where $[n]$ is the functor shifting the grading by $-n$.

A representation M is called *nilpotent* if $A^N = 0$ for $N \gg 0$. Let \mathbf{NR} denote the category of nilpotent representations of the quiver \mathbf{A}_∞ and \mathbf{NR}_n the category of nilpotent representations of the quiver $\widetilde{\mathbf{A}}_{n-1}$.

The dimension of a representation M of the quiver \mathbf{A}_∞ is just a sequence of nonnegative integers equal to the dimensions of the graded components of M . If M is a representation of the quiver $\widetilde{\mathbf{A}}_{n-1}$ then $\dim M$ is a n -periodic sequence.

Recall the well known classification of indecomposable objects of \mathbf{NR}_n . They are classified up to isomorphism by pairs (p, q) where $p \leq q$ are integers defined up

to simultaneous translation by a multiple of n . The representation corresponding to (p, q) is denoted by $M_{(p, q)}$. It has a basis e_p, e_{p+1}, \dots, e_q with e_j of degree $j \pmod n$, and we have $e_p \rightarrow e_{p+1} \rightarrow \dots \rightarrow e_q \rightarrow 0$ in the representation. We will denote the Grothendieck group $\mathbf{K}(\mathbf{NR}_n)$ by \mathbf{K}_n . It has a basis $[M_{(p, q)}]$, p, q as above. We will denote this basis by R^+ . An element of R^+ will be called *raiz*, and sometimes a raiz $[M_{(p, q)}]$ will be denoted simply by (p, q) . Given an integer s we can identify the set R^+ with the set

$$R_s^+ = \{(p, q) \mid p \leq q \text{ and } s \leq q \leq s + n - 1\}$$

A raiz (i, i) will be called *simple*, and sometimes will be denoted simply by i . We denote by $I \cong \mathbb{Z}/n\mathbb{Z}$ the set of all simple raiz.

Definition 2.1.1. We say that a raiz $\theta = (p, q) \in R^+$ begins (resp. ends) at a simple raiz i iff $i = p \pmod n$ (resp. $i = q \pmod n$). We denote by $\mathbf{B}_i \subset R^+ \cup \{0\}$ (resp. $\mathbf{E}_i \subset R^+ \cup \{0\}$) the set consisting of 0 and of all raiz beginning (resp. ending) at i .

Definition 2.1.2. Given two indecomposable representations θ, ϑ such that for some i we have $\theta \in \mathbf{B}_i$, $\vartheta \in \mathbf{E}_{i-1}$, there is a unique (isomorphism class of an) indecomposable representation η fitting into exact sequence

$$0 \rightarrow \theta \rightarrow \eta \rightarrow \vartheta \rightarrow 0$$

We say that $\eta = \vartheta * \theta$, $\vartheta = \eta/\theta$.

The dimension of a representation may be viewed as an element of the lattice Y of n -periodic sequences of integers. We identify a simple raiz $i \in I$ with the dimension of the corresponding simple representation, thus if $\alpha \in Y$ then $\alpha + i \in Y$ is the same as $\alpha + \dim i$.

The above identification gives rise to the identification $Y = \mathbb{Z}[I]$. Thus the dimension may be viewed as a map

$$\dim : \mathbf{K}_n \rightarrow \mathbb{N}[I] \subset \mathbb{Z}[I] = Y.$$

We will consider the following elements of the dual lattice Y^\vee . For $i \in \mathbb{Z}/n\mathbb{Z}$ we define

$$\langle i', y_\bullet \rangle = 2y_i - y_{i-1} - y_{i+1} \tag{1}$$

where $y_\bullet = (\dots, y_{-1}, y_0, y_1, \dots) \in Y$.

The $I \times I$ matrix

$$a_{ij} = \langle i', j \rangle = \begin{cases} 2, & \text{if } j = i \pmod n; \\ -1, & \text{if } j = i \pm 1 \pmod n \text{ and } n \neq 2; \\ -2, & \text{if } j = i + 1 \pmod n \text{ and } n = 2; \\ 0, & \text{if } j \neq i, i \pm 1 \pmod n. \end{cases}$$

is the affine Cartan matrix of type $\tilde{\mathbf{A}}_{n-1}$.

For $\alpha \in \mathbb{N}[I]$ we define $|\alpha|$ as the sum of all coordinates of α . For $\alpha, \beta \in \mathbb{N}[I]$ we say that $\alpha \leq \beta$ iff $\beta - \alpha \in \mathbb{N}[I]$.

Consider also the lattice $Y^{(2)} \supset Y$ of sequences of integers n -periodic modulo linear term, that is

$$Y^{(2)} = \{y_\bullet \mid y_{p+n} - y_p = ap + b \text{ for some } a, b \in \mathbb{Z}\}.$$

Also, we define

$$\rho = (\dots, \rho_{-1}, \rho_0, \rho_1, \dots) \in Y^{(2)} \otimes \mathbb{Q}, \quad \rho_p = -p^2/2.$$

Lemma 2.1.3. *The functional i' defined by (1) is well defined on the lattice $Y^{(2)}$ (that is $\langle i', y_\bullet \rangle$ depends only on $i \bmod n$ when $y_\bullet \in Y^{(2)}$). Moreover, we have*

$$\langle i', \rho \rangle = 1$$

for all $i \in I$.

Proof. Evident. □

2.2. Partitions. Assume that we have a set X and a function $\xi : X \rightarrow (\mathbb{N}[I] - \{0\})$. For any $\alpha \in \mathbb{N}[I]$ we define an ordered m -terms partition \tilde{A} of α with respect to (X, ξ) as a map $\tilde{A} : [m] = \{1, \dots, m\} \rightarrow X$ such that

$$\sum_{p=1}^m \xi(\tilde{A}(p)) = \alpha.$$

We denote the set of all ordered m -terms partitions of α with respect to (X, ξ) by $\tilde{P}_{X, \xi}^m(\alpha)$. The group of permutations S_m acts naturally on the set $\tilde{P}_{X, \xi}^m(\alpha)$. We denote by $S_{\tilde{A}} \subset S_m$ the stabilizer subgroup of $\tilde{A} \in \tilde{P}_{X, \xi}^m(\alpha)$.

We define an unordered m -terms partition (or, more simply, a partition) A of α with respect to (X, ξ) as an S_m -orbit in the set $\tilde{P}_{X, \xi}^m(\alpha)$, and denote by

$$P_{X, \xi}(\alpha) = \bigsqcup_{m=0}^{\infty} P_{X, \xi}^m(\alpha) = \bigsqcup_{m=0}^{\infty} \tilde{P}_{X, \xi}^m(\alpha) / S_m$$

the set of all partitions of α with respect to (X, ξ) .

Given a partition A we denote by \tilde{A} its *ordering*, that is any representative of A in the set of ordered partitions. Let $\alpha_p = \tilde{A}(p)$ ($p = 1, \dots, m$). We will denote the partition A by $\{\{\alpha_1, \dots, \alpha_m\}\}$, and for $A = \{\{\alpha_1, \dots, \alpha_m\}\}$ we will denote

$$|A| = \xi(\alpha_1) + \dots + \xi(\alpha_m) \in \mathbb{N}[I], \quad \mathsf{K}(A) = m,$$

and

$$m(\alpha, A) = \#\{p \in [m] \mid \alpha_p = \alpha\}.$$

Thus, we always have $A \in P_{X, \xi}(|A|)$.

We will use the following types of partitions.

Usual partitions: Here we put $X = \mathbb{N}[I] - \{0\}$, $\xi = \mathbf{id}$. We denote the set of usual partitions of α by $\Gamma(\alpha)$.

Kostant partitions: Here we put $X = R^+$, $\xi = \dim$. We denote the set of Kostant partitions of α by $\mathfrak{K}(\alpha)$.

Multipartitions: Here we put $X = \bigsqcup_{\gamma \in (\mathbb{N}[I] - \{0\})} \mathfrak{K}(\gamma)$, $\xi(\kappa) = |\kappa|$. We denote the set of multipartitions of α by $\mathfrak{M}(\alpha)$.

Note that if $\theta \in R^+$ is a raiz, then the set $\mathfrak{K}(\dim \theta)$ of Kostant partitions of $\dim \theta$ contains an element $\{\{\theta\}\}$. Such Kostant partition is called a *simple Kostant partition*. A multipartition $\mu = \{\{\kappa_1, \dots, \kappa_m\}\}$ is called a *simple multipartition* if all κ_p are simple Kostant partitions.

We have the following natural maps: $\dim : \mathfrak{K}(\alpha) \rightarrow \Gamma(\alpha)$, $\mathfrak{K}(\alpha) \hookrightarrow \mathfrak{M}(\alpha)$ (the set of Kostant partitions of α is identified with the set of simple multipartitions of α), and the projection $|\cdot| : \mathfrak{M}(\alpha) \rightarrow \Gamma(\alpha)$ (we define $|\{\kappa_1, \dots, \kappa_m\}| := \{|\kappa_1|, \dots, |\kappa_m|\}\}$).

2.3. Configuration spaces. Let C^α denote the configuration space of effective divisors on the curve C with coefficients in $\mathbb{N}[I]$ of degree α . If $\alpha = \sum_{i \in I} a_i i$ then the space C^α is isomorphic to the product of the symmetric powers of C , more precisely

$$C^\alpha \cong \prod_{i \in I} S^{a_i} C.$$

The space C^α carries the natural diagonal stratification, the strata of which are in one-to-one correspondence with partitions of α :

$$C^\alpha = \bigsqcup_{\Gamma \in \Gamma(\alpha)} C_\Gamma^\alpha.$$

The stratum C_Γ^α , corresponding to the ordered partition $\tilde{\Gamma} = \{\{\alpha_1, \dots, \alpha_m\}\}$ consists of all divisors $\sum_{p=1}^m \alpha_p x_p$, where x_1, \dots, x_p are pairwise distinct points of C . Thus we have an isomorphism

$$C_\Gamma^\alpha \cong (C^m - \Delta)/S_{\tilde{\Gamma}},$$

where $\Delta \subset C^m$ is the big diagonal.

If $\mu \in \mathfrak{M}(\alpha)$ is a multipartition and $\Gamma = |\mu| \in \Gamma(\alpha)$ then it is clear that $S_{\tilde{\mu}} \subset S_{\tilde{\Gamma}}$. Let C_μ^α denote the quotient $(C^m - \Delta)/S_{\tilde{\mu}}$. The space C_μ^α is a $\frac{|S_{\tilde{\Gamma}}|}{|S_{\tilde{\mu}}|}$ -fold covering of the space C_Γ^α .

2.4. Nilpotent $\tilde{\mathbf{A}}_{n-1}$ -modules over C . Let $\alpha \in \mathbb{N}[I]$. Recall that the isomorphism classes of α -dimensional objects of the category \mathbf{NR}_n are in one-to-one correspondence with Kostant partitions of α . We denote by $\kappa(M) \in \mathfrak{K}(\dim M)$ the Kostant partition corresponding to the isomorphism class of M . We denote by M_θ an indecomposable representation corresponding to a raiz $\theta \in R^+$, and we denote by M_κ a representation corresponding to a Kostant partition $\kappa \in \mathfrak{K}(\alpha)$.

Let $\mathbf{i} : C \rightarrow S$ be a closed embedding of a smooth curve C into a surface S . Let $\mathbf{NR}_n(C)$ denote the category of nilpotent representations of the quiver $\tilde{\mathbf{A}}_{n-1}$ in the category of cohernet sheaves on S with 0-dimensional support on the curve C . Every object T of the category $\mathbf{NR}_n(C)$ can be decomposed as $T = \bigoplus_{x \in C} T_x$, where T_x is an object concentrated at a point $x \in C$. Let

$$\Gamma, \Gamma_x : \mathbf{NR}_n(C) \rightarrow \mathbf{NR}_n, \quad \Gamma(T) = \Gamma(S, T), \quad \Gamma_x(T) = \Gamma(S, T_x)$$

denote the functor of global sections and of global sections with support at x respectively.

Let κ_x be the Kostant partition, corresponding to the isomorphism class of the object $\Gamma_x(T)$ of the category \mathbf{NR}_n . Then the nontrivial Kostant partitions κ_x , $x \in C$ form a multipartition $\mu(T) \in \mathfrak{M}(\alpha)$ of $\alpha = \dim \Gamma(T)$. We call the objects T and T' of $\mathbf{NR}_n(C)$ *equivalent* if $\mu(T) = \mu(T')$.

Thus the set of equivalence classes of objects of the category $\mathbf{NR}_n(C)$ are in one-to-one correspondence with multipartitions.

3. THE SPACE K_α

3.1. Definition and piecification. We fix a smooth surface S and a smooth curve $C \subset S$. Let $\overset{\circ}{S} = S - C$ be the complement, $\mathbf{j} : \overset{\circ}{S} \rightarrow S$ the open embedding, and $\mathbf{i} : C \rightarrow S$ the closed embedding. Let $[C] \in H^2(S, \mathbb{Z})$ be the fundamental class of the marked curve $C \subset S$. We denote by $d = [C]^2 = \deg \mathcal{N}_{C/S}$ the degree of the normal bundle and by $g = g(C)$ the genus of the curve C .

Let V be a n -dimensional vector space. We fix a flag

$$\cdots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset V \otimes \mathbf{j}_* \mathcal{O}_{\overset{\circ}{S}}$$

of rank n vector bundles on the surface S such that

$$\begin{aligned} \mathcal{E}_{p-n} &= \mathcal{E}_p(-C) = \mathcal{E}_p \otimes \mathcal{O}_S(-C) \subset \mathcal{E}_p && \text{(periodicity)} \\ c_1(\mathcal{E}_{p+1}) &= c_1(\mathcal{E}_p) + [C] && \text{(normalization)} \end{aligned}$$

It follows that $\mathcal{E}_p/\mathcal{E}_{p-1} = \mathbf{i}_* \mathcal{L}_p$, where \mathcal{L}_p are line bundles on the curve C . Moreover, the periodicity implies that

$$\mathcal{L}_{p+n} = \mathcal{L}_p \otimes \mathbf{i}^* \mathcal{O}_S(C) = \mathcal{L}_p \otimes \mathcal{N}_{C/S}$$

Let α_0 denote the sequence formed by $-ch_2(\mathcal{E}_p)$, where $ch_2 = c_1^2/2 - c_2$ is the second coefficient of the Chern character.

Lemma 3.1.1. *We have $\alpha_0 \in Y^{(2)} \otimes \mathbb{Q}$ and $\langle i', \alpha_0 \rangle = \deg \mathcal{L}_{i+1} - \deg \mathcal{L}_i$. Moreover, we have $\sum_{i \in I} \langle i', \alpha_0 \rangle = d$.*

Proof. We have

$$ch(\mathcal{E}_{p+n}) - ch(\mathcal{E}_p) = ch(\mathcal{E}_p)(1 + [C] + [C]^2/2) - ch(\mathcal{E}_p) = ch(\mathcal{E}_p)([C] + d/2[\text{point}]),$$

hence

$$ch_2(\mathcal{E}_{p+n}) - ch_2(\mathcal{E}_p) = ch_1(\mathcal{E}_p) \cdot [C] + ch_0(\mathcal{E}_p) \cdot d/2 = pd + c_1(\mathcal{E}_0) \cdot [C] + nd/2,$$

hence $\alpha_0 \in Y^{(2)} \otimes \mathbb{Q}$.

On the other hand by the Riemann-Roch-Grothendick Theorem we have

$$ch(\mathbf{i}_* \mathcal{L}_p) = [C] + (\deg \mathcal{L}_p - d/2)[\text{point}],$$

hence

$$\deg \mathcal{L}_p = ch_2(\mathcal{E}_p) - ch_2(\mathcal{E}_{p-1}) + d/2$$

hence

$$\deg \mathcal{L}_{p+1} - \deg \mathcal{L}_p = ch_2(\mathcal{E}_{p+1}) + ch_2(\mathcal{E}_{p-1}) - 2ch_2(\mathcal{E}_p) = \langle p', \alpha_0 \rangle.$$

Finally, we have

$$\sum_{i \in I} \langle i', \alpha_0 \rangle = \sum_{p=0}^{n-1} (\deg \mathcal{L}_{p+1} - \deg \mathcal{L}_p) = \deg \mathcal{L}_n - \deg \mathcal{L}_0 = d.$$

□

Any infinite flag of coherent sheaves on the surface S can be considered as a representation of the quiver \mathbf{A}_∞ . Given a periodic subflag $E_\bullet \subset \mathcal{E}_\bullet$, such that $E_\bullet|_S \cong \mathcal{E}_\bullet|_S$ we denote by $T_\bullet = \mathcal{E}_\bullet/E_\bullet$ the quotient representation of the quiver \mathbf{A}_∞ . Assume that the support of T_\bullet is 0-dimensional. Then choosing a trivialization of the normal bundle $\mathcal{N}_{C/S}$ in a neighbourhood of $\text{supp } T_\bullet$ we obtain an isomorphism

$$T_{p+n} \cong T_n,$$

hence T_\bullet can be considered as a representation of the cyclic quiver $\tilde{\mathbf{A}}_{n-1}$ in the category of coherent sheaves on the curve C . Then $\Gamma(T_\bullet)$ is a representation of $\tilde{\mathbf{A}}_{n-1}$ in the category of vector spaces. It is clear that both T_\bullet and $\Gamma(T_\bullet)$ are nilpotent.

Definition 3.1.2. Let $K_\alpha(\mathcal{E}_\bullet)$ denote the space of all periodic subflags $E_\bullet \subset \mathcal{E}_\bullet$ such that T_\bullet is an object of the category $\mathbf{NR}_n(C)$ and $\dim \Gamma(T_\bullet) = \alpha$.

We will denote the space $K_\alpha(\mathcal{E}_\bullet)$ simply by K_α for brevity.

For a multipartition $\mu(E_\bullet) \in \mathfrak{M}(\alpha)$ let $K_\mu \subset K_\alpha$ denote the subspace of all E_\bullet such that the equivalence class $\mu(T_\bullet)$ of the object $T_\bullet = \mathcal{E}_\bullet/E_\bullet$ of the category $\mathbf{NR}_n(C)$ is equal to μ . This defines a piecification

$$K_\alpha = \bigsqcup_{\mu \in \mathfrak{M}(\alpha)} K_\mu \quad (2)$$

It is clear that the equivalence class $\mu(T_\bullet)$ doesn't depend on the choice of the trivialization of the normal bundle $\mathcal{N}_{C/S}$ involved.

Let $\kappa_x(T)$ denote the isomorphism class of the representation $\Gamma_x(T)$. Then by definition the multipartition $\mu(T_\bullet)$ is formed by nontrivial Kostant partitions $\kappa_x(T_\bullet)$. Hence we have a map

$$\sigma : K_\mu \rightarrow C_\mu^\alpha, \quad E_\bullet \mapsto \sum_{x \in C} \kappa_x(\mathcal{E}_\bullet/E_\bullet) x.$$

Let $\mu = \{\{\kappa_1, \dots, \kappa_m\}\}$. Given an element $\sum \kappa_r x_r \in C_\mu^\alpha$ let $F_\mu(\sum \kappa_r x_r)$ denote the fiber $\sigma^{-1}(\sum \kappa_r x_r) \subset K_\mu$.

Lemma 3.1.3. *The map σ is a locally trivial fibration. Moreover, we have an isomorphism*

$$F_\mu(\kappa_1 x_1 + \dots + \kappa_m x_m) = F_{\{\{\kappa_1\}\}}(\kappa_1 x_1) \times \dots \times F_{\{\{\kappa_m\}\}}(\kappa_m x_m).$$

Proof. Evident. □

Thus the description of the stratum $K_\mu \subset K_\alpha$ reduces to the description of the space $F_{\{\{\kappa\}\}}(\kappa x)$ which is called a *simple fiber*.

3.2. Simple fiber. This subsection is devoted to the proof of the following Theorem. We fix a point $x \in C$ and a Kostant partition κ . We denote the simple fiber $F_{\{\{\kappa\}\}}(\kappa x)$ by F_κ for brevity.

Recall that

$$F_\kappa = \{E_\bullet \subset \mathcal{E}_\bullet \mid \text{supp}(\mathcal{E}_\bullet/E_\bullet) = \{x\} \text{ and } \kappa(\Gamma(\mathcal{E}_\bullet/E_\bullet)) = \kappa\}$$

Theorem 1. *The space F_κ is a pseudoaffine space of dimension $||\kappa|| - K(\kappa)$.*

It is clear that the space F_κ depends only on the local properties of the surface S near the point x . So, in this subsection we can and will replace S by a small neighbourhood of x . This allows to fix a trivialization of the normal bundle $\mathcal{N}_{C/S}$, giving an isomorphisms

$$\mathcal{L}_{p+n} = \mathcal{L}_p.$$

We fix some integer s .

Let κ_p^q ($p \leq q$, $s \leq q \leq s+n-1$) be the coordinates of the Kostant partition κ with respect to R_s^+ .

For any collection κ_p^q and subsets $I, J \subset \mathbb{Z}$ we define

$$\kappa_I^J = \sum_{p \in I, q \in J} \kappa_p^q.$$

Another possible definition of the collection κ_p^q is given by the following Lemma.

Lemma 3.2.1. *Assume that N is a n -periodic representation of the quiver \mathbf{A}_∞ such that its isomorphism class in the category of representations of the quiver $\tilde{\mathbf{A}}_{n-1}$ is equal to κ . Let \hat{N}_p denote the kernel of the map $N_p \rightarrow N_{s+n}$. Then for all $p \leq q \leq s+n-1$ we have*

$$\text{rank}(\hat{N}_p \rightarrow \hat{N}_q) = \kappa_{\leq p}^{\geq q}.$$

Proof. The Lemma follows from the calculation of the contributions of the summands M_θ ($\theta = (p', q') \in R_s^+$) of N into the rank of the above map. It suffices to note that M_θ contributes to the rank of the map iff $p' \leq p$ and $q \leq q'$. \square

Assume that $E_\bullet \in F_\kappa$ and let $T_\bullet = \mathcal{E}_\bullet / E_\bullet$.

Let us denote the torsion sheaf $L^1 \mathbf{i}^* T_s$ by R . Now we will introduce a pair of filtrations on R .

The first of them can be constructed quite easily.

Lemma 3.2.2. *There are natural isomorphisms*

$$\mathbf{i}^* T_s = \mathbf{Coker}(T_{s-n} \rightarrow T_s), \quad L^1 \mathbf{i}^* T_s = \mathbf{Ker}(T_s \rightarrow T_{s+n}).$$

Proof. Evident. \square

Let us denote

$$R_i = \mathbf{Ker}(T_s \rightarrow T_{s+i}) \quad (0 \leq i \leq n).$$

This defines the *right* filtration

$$0 = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = R$$

of the sheaf R .

The second one is a little bit more complicated. First we consider a filtration on the sheaf T_s formed by the sheaves

$$T_i^s = \mathcal{E}_i / (E_s \cap \mathcal{E}_i) = \mathbf{Im}(T_i \rightarrow T_s) \subset T_s.$$

Then we consider

$$R^i = L^1 \mathbf{i}^* (\mathcal{E}_i / (E_s \cap \mathcal{E}_i)) = L^1 \mathbf{i}^* T_i^s \subset L^1 \mathbf{i}^* T_s = R.$$

This defines the *left* filtration

$$\cdots \subset R^{s-2} \subset R^{s-1} \subset R^s = R.$$

of the sheaf R .

Remark 3.2.3. Note that the left filtration R^\bullet of the sheaf R is defined by the subsheaf $E_s \subset \mathcal{E}_s$ only.

Lemma 3.2.4. *We have*

$$\text{rank}(\Gamma(R^p) \rightarrow \Gamma(R) \rightarrow \Gamma(R/R_i)) = \kappa_{\leq p}^{\geq s+i}, \quad (p \leq s, 0 \leq i < n). \quad (3)$$

Proof. Let $N_\bullet = \Gamma(T_\bullet)$. Then according to the Lemma 3.2.1 we have $\kappa_{\leq p}^{\geq s+i} = \text{rank}(\hat{N}_p \rightarrow \hat{N}_{s+i})$. Since $\Gamma(\bullet)$ is an exact functor on the category of torsion sheaves on the curve it follows that

$$\hat{N}_p = \Gamma\left(\frac{\mathcal{E}_p \cap E_s(C)}{E_p}\right), \quad \hat{N}_{s+i} = \Gamma\left(\frac{\mathcal{E}_{s+i} \cap E_s(C)}{E_{s+i}}\right).$$

It is evident that

$$\text{Im}\left(\frac{\mathcal{E}_p \cap E_s(C)}{E_p} \rightarrow \frac{\mathcal{E}_{s+i} \cap E_s(C)}{E_{s+i}}\right) = \frac{\mathcal{E}_p \cap E_s(C)}{\mathcal{E}_p \cap E_{s+i}},$$

hence

$$\dim \Gamma\left(\frac{\mathcal{E}_p \cap E_s(C)}{\mathcal{E}_p \cap E_{s+i}}\right) = \kappa_{\leq p}^{\geq s+i} \quad \text{for all } p \leq s, 0 \leq i < n. \quad (4)$$

On the other hand, it is easy to show that

$$R^p = \frac{\mathcal{E}_p \cap E_s(C)}{\mathcal{E}_p \cap E_s}, \quad R_i = \frac{\mathcal{E}_s \cap E_{s+i}}{E_s} \quad \text{hence} \quad R/R_i = \frac{\mathcal{E}_s \cap E_s(C)}{\mathcal{E}_s \cap E_{s+i}}.$$

and the image of the map $R^p \rightarrow R/R_i$ is equal to

$$\frac{\mathcal{E}_p \cap E_s(C)}{\mathcal{E}_p \cap E_{s+i}}$$

and the Lemma follows. \square

For every $t \leq s$ let X_κ^t denote the space of all subsheaves $E \subset \mathcal{E}_t$ with a filtration

$$0 = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = R = L^1 \mathbf{i}^*(\mathcal{E}_t/E)$$

such that the quotient \mathcal{E}_t/E is concentrated at the point x and the filtration R_\bullet together with the left filtration $R^p := L^1 \mathbf{i}^*(\mathcal{E}_p/(E \cap \mathcal{E}_p))$ satisfy the condition

$$\text{rank}(\Gamma(R^p) \rightarrow \Gamma(R) \rightarrow \Gamma(R/R_i)) = \kappa_{\leq p}^{\geq s+i}, \quad (p \leq t, 0 \leq i < n). \quad (5)$$

We have an obvious map $\pi_s : F_\kappa \rightarrow X_\kappa^s$ sending E_\bullet to (E_s, R_\bullet) , where R_\bullet is the right filtration of the sheaf R .

Thus the problem of description of the space F_κ reduces to the description of the space X_κ^s and to the description of the fiber of the map π_s .

We begin with some notation. Assume that $(E, R_\bullet) \in X_\kappa^t$ for some $t \leq s$. We denote the intersection $\mathcal{E}_i \cap E$ by E_i^t and the quotient \mathcal{E}_i/E_i^t by T_i^t . Then we have a filtration

$$\cdots \subset T_{t-2}^t \subset T_{t-1}^t \subset T_t^t =: T.$$

Lemma 3.2.5. *We have $\mathbf{i}^* T_i^t = T_i^t / T_{i-n}^t$ for all $i \leq t$.*

Proof. Since $T_i^t = \mathcal{E}_i/E_i^t$, it follows that $\mathbf{i}^* T_i^t$ is isomorphic to the cokernel of the map $T_i^t \rightarrow T_i^t$, induced by the embeddings $\mathcal{E}_i(-C) = \mathcal{E}_{i-n} \subset \mathcal{E}_i$ and $E_i^t(-C) \subset E_i^t$. However, since we have

$$E_i^t(-C) \subset E_{i-n}^t \subset \mathcal{E}_{i-n}$$

it follows that the morphism $T_i^t \rightarrow T_i^t$ factors as the composition of the surjection $T_i^t \rightarrow T_{i-n}^t$ and the embedding $T_{i-n}^t \rightarrow T_i^t$. The Lemma follows. \square

Lemma 3.2.6. *We have*

$$\dim \Gamma(T_p^t) = \sum_{r=0}^{\infty} \kappa_{\leq p-rn}^{\geq s}.$$

Proof. Follows from 3.2.5 and (5) since $\dim \Gamma(\mathbf{i}^* T_i^t) = \dim \Gamma(L^1 \mathbf{i}^* T_i^t) = \dim \Gamma(R^i)$. \square

Consider the restriction of the embedding $\mathcal{E}_{i-n} \rightarrow \mathcal{E}_{i-1}$ to the curve C . Since $\mathbf{i}^* \mathcal{E}_{i-1} = \mathcal{E}_{i-1}/\mathcal{E}_{i-n-1}$ it follows that the map $\mathbf{i}^* \mathcal{E}_{i-n} \rightarrow \mathbf{i}^* \mathcal{E}_{i-1}$ factors as $\mathbf{i}^* \mathcal{E}_{i-n} \rightarrow \mathcal{L}_{i-n} \rightarrow \mathbf{i}^* \mathcal{E}_{i-1}$. Thus, we have an embedding $\mathcal{L}_i = \mathcal{L}_{i-n} \rightarrow \mathbf{i}^* \mathcal{E}_{i-1}$.

Lemma 3.2.7. *The kernel of the composition $\mathcal{L}_i \rightarrow \mathbf{i}^* \mathcal{E}_{i-1} \rightarrow \mathbf{i}^* T_{i-1}^t$ is equal to $\mathcal{L}_i(-\sum_{r=1}^{\infty} \kappa_{i-rn}^{\geq s} x)$.*

Proof. It suffices to show that the image T of the above map is a torsion sheaf with $\dim \Gamma(T) = \kappa_{\leq i-n}^{\geq s}$. Note that the commutative diagram

$$\begin{array}{ccc} \mathbf{i}^* \mathcal{E}_{i-n} & \longrightarrow & \mathbf{i}^* \mathcal{E}_{i-1} \\ \downarrow & & \downarrow \\ \mathbf{i}^* T_{i-n}^t & \longrightarrow & \mathbf{i}^* T_{i-1}^t \end{array}$$

implies $T = \mathbf{Im}(\mathbf{i}^* T_{i-n}^t \rightarrow \mathbf{i}^* T_{i-1}^t)$. From the Lemma 3.2.5 it follows that

$$T = \mathbf{Im}(T_{i-n}^t/T_{i-2n}^t \rightarrow T_{i-1}^t/T_{i-n-1}^t) = T_{i-n}^t/T_{i-n-1}^t,$$

hence according to the Lemma 3.2.6 we have

$$\dim \Gamma(T) = \dim \Gamma(T_{i-n}^t) - \dim \Gamma(T_{i-n-1}^t) = \sum_{r=1}^{\infty} \kappa_{i-rn}^{\geq s} - \sum_{r=1}^{\infty} \kappa_{i-rn-1}^{\geq s} = \sum_{r=1}^{\infty} \kappa_{i-rn}^{\geq s}.$$

\square

Corollary 3.2.8. *The composition*

$$\mathcal{L}_i \left(- \sum_{r=0}^{\infty} \kappa_{i-rn}^{\geq s} x \right) \rightarrow \mathcal{L}_i \rightarrow \mathbf{i}^* \mathcal{E}_{i-1} \rightarrow \mathbf{i}^* T_{i-1}^t$$

vanishes.

Proof. Follows from the above Lemma. \square

Consider the map $\varpi_t : X_{\kappa}^t \rightarrow X_{\kappa}^{t-1}$ given by

$$\varpi_t(E, R_{\bullet} \subset R^t) = (E \cap \mathcal{E}_{t-1}, (R^{t-1} \cap R_{\bullet}) \subset R^{t-1}).$$

Proposition 3.2.9. *The map ϖ_t is a locally trivial pseudoaffine fibration with the fiber $\varpi_t^{-1}(E', R'_{\bullet})$ being a pseudoaffine space of dimension $\kappa_{\leq t-1}^{\geq s}$.*

The proof of the Proposition 3.2.9 consists of a few steps.

Consider the space Y of all pairs (E, R'_{\bullet}) , where E is a subsheaf in \mathcal{E}_t and R'_{\bullet} is a filtration in the sheaf $R' = L^1 \mathbf{i}^*(\mathcal{E}_{t-1}/(E \cap \mathcal{E}_{t-1}))$ such that \mathcal{E}_t/E is concentrated at x and $\dim \Gamma(\mathcal{E}_t/E) = \sum_{r=0}^{\infty} \kappa_{\leq t-rn}^{\geq s}$ and for all $p \leq t-1$ the conditions (5) are satisfied.

We have natural maps $\varpi'_t : X_{\kappa}^t \rightarrow Y$, $(E, R_{\bullet}) \mapsto (E, R_{\bullet} \cap R')$, and $\xi : Y \rightarrow X_{\kappa}^{t-1}$, $(E, R'_{\bullet}) \mapsto (E \cap \mathcal{E}_{t-1}, R'_{\bullet})$, and evidently $\varpi_t = \xi \cdot \varpi'_t$.

We begin with the description of the space Y .

Proposition 3.2.10. *The space Y is a torsor over the vector bundle*

$$H := \mathbf{Hom}_{X_\kappa^{t-1}}(\mathcal{L}_t(-kx), R'),$$

where $k = \sum_{r=0}^{\infty} \kappa_{t-rn}^{\geq s}$.

Proof. Let $(E', R'_\bullet) \in X_\kappa^{t-1}$ and let T' denote the quotient \mathcal{E}_{t-1}/E' . If E is a subsheaf in \mathcal{E}_t such that $E \cap \mathcal{E}_{t-1} = E'$ and $(E, R'_\bullet) \in Y$, then E/E' is a subsheaf in $\mathcal{E}_t/\mathcal{E}_{t-1} = \mathbf{i}_*\mathcal{L}_t$, and

$$\dim \Gamma \left(\frac{\mathcal{E}_t/\mathcal{E}_{t-1}}{E/E'} \right) = \dim \Gamma \left(\frac{\mathcal{E}_t/E}{\mathcal{E}_{t-1}/E'} \right) = \sum_{r=0}^{\infty} \kappa_{\leq t-rn}^{\geq s} - \sum_{r=0}^{\infty} \kappa_{\leq t-1-rn}^{\geq s} = k,$$

hence $E/E' = \mathbf{i}_*\mathcal{L}_t(-kx)$. Let \widetilde{W} denote the quotient $\mathcal{L}_t/\mathcal{L}_t(-kx)$, and let $\widetilde{\mathcal{E}}$ denote the kernel of the composition $\mathcal{E}_t \rightarrow \mathbf{i}_*\mathcal{L}_t \rightarrow \widetilde{W}$. It follows that Y is the space of all extensions of the projection $\mathcal{E}_{t-1} \rightarrow T'$ to the map $\mathcal{E}_{t-1} \subset \widetilde{\mathcal{E}} \rightarrow T'$. It follows from [7] that we have the obstruction map $\eta : X_\kappa^{t-1} \rightarrow \mathbf{Ext}^1(\mathbf{i}_*\mathcal{L}_t(-kx), T')$ and that Y is a $\mathbf{Hom}(\mathbf{i}_*\mathcal{L}_t(-kx), T')$ -torsor over the zero locus of the map η . Note that

$$\mathbf{Hom}(\mathbf{i}_*\mathcal{L}_t(-kx), T') \cong \mathbf{Hom}(\mathcal{L}_t(-kx), L^1\mathbf{i}^*T') = H.$$

Hence it suffices to prove that $\eta = 0$ in our case.

To this end note that the obstruction $\eta(E')$ is equal to the Ioneda product of the embedding $\mathbf{i}_*\mathcal{L}_t(-kx) \rightarrow \mathbf{i}_*\mathcal{L}_t$, of the extension

$$0 \rightarrow \mathcal{E}_{t-1} \rightarrow \mathcal{E}_t \rightarrow \mathbf{i}_*\mathcal{L}_t \rightarrow 0, \quad (6)$$

and of the projection $\mathcal{E}_{t-1} \rightarrow T'$. On the other hand, we have an isomorphism

$$\mathbf{Ext}^1(\mathbf{i}_*\mathcal{L}_t, \mathcal{E}_{t-1}) \cong \mathbf{Hom}(\mathcal{L}_t, \mathbf{i}^*\mathcal{E}_{t-1})$$

under which the extension (6) corresponds to the natural embedding $\mathcal{L}_t \rightarrow \mathbf{i}^*\mathcal{E}_{t-1}$ (see Lemma 3.2.7). Hence the Corollary 3.2.8 implies the vanishing of the obstruction. \square

Assume that $(E, R'_\bullet) \in Y$ and let $E' = E \cap \mathcal{E}_{t-1}$. Let $T = \mathcal{E}_t/E$ and $T' = \mathcal{E}_{t-1}/E' \subset T$. It follows from the proof of the Proposition 3.2.10 that we have an exact sequence

$$0 \rightarrow T' \rightarrow T \rightarrow \mathbf{i}_*(\mathcal{L}_t/\mathcal{L}_t(-kx)) \rightarrow 0,$$

hence R/R' is a subsheaf in $L^1\mathbf{i}^*(\mathbf{i}_*(\mathcal{L}_t/\mathcal{L}_t(-kx))) = \mathcal{L}_t/\mathcal{L}_t(-kx)$. On the other hand we have $\dim \Gamma(R') = \kappa_{\leq t-1}^{\geq s}$ by the definition of the space Y and

$$\begin{aligned} \dim \Gamma(R) &= \dim \Gamma(L^1\mathbf{i}^*T) = \dim \Gamma(\mathbf{i}^*T) = \dim \Gamma(T/T_{t-n}^t) = \\ &= \dim \Gamma(T) - \dim \Gamma(T_{t-n}^t) = \dim \Gamma(T) - \dim \Gamma(T_{t-n}^{t-1}) = \kappa_{\leq t}^{\geq s}. \end{aligned}$$

Hence $\dim \Gamma(R/R') = \kappa_{\leq t}^{\geq s} - \kappa_{\leq t-1}^{\geq s} = \kappa_t^{\geq s}$. This means that

$$R/R' = W := \mathcal{L}_t((\kappa_t^{\geq s} - k)x)/\mathcal{L}_t(-kx) = \mathcal{L}_t((\kappa_t^{[s,s+n-1]} - k)x)/\mathcal{L}_t(-kx).$$

Now, if R_\bullet is a filtration on R such that $(E, R_\bullet) \in X_\kappa^t$ then we have $R_i/R'_i \subset R/R' = W$ and

$$\dim \Gamma(R_i/R'_i) = \dim \Gamma(R_i) - \dim \Gamma(R'_i) = \kappa_{\leq t}^{[s,s+i-1]} - \kappa_{\leq t-1}^{[s,s+i-1]} = \kappa_t^{[s,s+i-1]},$$

hence

$$R_i/R'_i = W_i := \mathcal{L}_t((\kappa_t^{[s,s+i-1]} - k)x)/\mathcal{L}_t(-kx).$$

Thus we have the following standard commutative diagram:

$$\begin{array}{ccccccc}
 0 & & 0 & & 0 & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 R'_\bullet & & R_\bullet & & W_\bullet & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 \longrightarrow R' \longrightarrow R \longrightarrow W \longrightarrow 0 & & & & & & (7) \\
 \downarrow \text{can} & & \downarrow & & \downarrow & & \\
 0 \longrightarrow R'/R'_\bullet \longrightarrow R/R_\bullet \longrightarrow W/W_\bullet \longrightarrow 0 & & & & & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
 \end{array}$$

Remark 3.2.11. This is a diagram in the category of representations of the quiver

$$\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \dots \longrightarrow \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet}$$

in the category of coherent sheaves on the curve C . From now and till the end of the proof of the Proposition the index \bullet indicates an object of this category. It will be rather important below that indices start from 0.

Let \tilde{R}_\bullet denote the kernel of the composition $R \rightarrow W \rightarrow W/W_\bullet$. From the standard technique of [7] it follows that

$$X_\kappa^t = \left\{ f \in \mathbf{Hom}(\tilde{R}_\bullet, R'/R'_\bullet) \mid \text{such that the triangle } \begin{array}{ccc} R' & \xrightarrow{\text{can}} & \tilde{R}_\bullet \\ \downarrow & \nearrow f & \\ R'/R'_\bullet & & \end{array} \text{ commutes} \right\}$$

In other words, we have the following cartesian square

$$\begin{array}{ccc}
 X_\kappa^t & \longrightarrow & \mathbf{Hom}_Y(\tilde{R}_\bullet, R'/R'_\bullet) \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{1 \times \text{can}} & Y \times \mathbf{Hom}_{X_\kappa^{t-1}}(R', R'/R'_\bullet)
 \end{array}$$

Consider the map $\varepsilon : Y \rightarrow \mathbf{Ext}^1(W, R')$ given by the middle row of (7) and the projection $\mathbf{Ext}^1(W, R') \rightarrow \mathbf{Ext}^1(W_\bullet, R')$ induced by the embedding $W_\bullet \rightarrow W$. Note that applying the functor $\mathbf{Hom}(\bullet, R')$ to the sequences

$$0 \rightarrow \mathcal{L}_t(-kx) \rightarrow \mathcal{L}_t((\kappa_t^{[s,s+n-1]} - k)x) \rightarrow W \rightarrow 0, \quad (8)$$

$$0 \rightarrow \mathcal{L}_t(-kx) \rightarrow \mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x) \rightarrow W_\bullet \rightarrow 0 \quad (9)$$

we obtain the morphisms of X_κ^{t-1} -spaces

$$\mathbf{Hom}(\mathcal{L}_t(-kx), R') \rightarrow \mathbf{Ext}^1(W, R') \quad \text{and} \quad \mathbf{Hom}(\mathcal{L}_t(-kx), R') \rightarrow \mathbf{Ext}^1(W_\bullet, R')$$

which define a natural fiberwise (over X_κ^{t-1}) action of the vector bundle H on $\mathbf{Ext}^1(W, R')$ and $\mathbf{Ext}^1(W_\bullet, R')$.

Lemma 3.2.12. *The map $\tilde{\varepsilon} : Y \xrightarrow{\varepsilon} \mathbf{Ext}^1(W, R') \rightarrow \mathbf{Ext}^1(W_\bullet, R')$ commutes with the action of H .*

Proof. Let \widetilde{W} denote the quotient $\mathcal{L}_t/\mathcal{L}_t(-kx)$. Recall that we have the following commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{\mathcal{E}} & \longrightarrow & \mathcal{E}_t & \longrightarrow & \mathbf{i}_*\widetilde{W} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbf{i}_*\mathcal{L}_t(-kx) & \longrightarrow & \mathbf{i}_*\mathcal{L}_t & \longrightarrow & \mathbf{i}_*\widetilde{W} \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}_t(-kx) & \longrightarrow & \mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x) & \longrightarrow & W_\bullet \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{L}_t(-kx) & \longrightarrow & \mathcal{L}_t((\kappa_t^{[s,s+n-1]} - k)x) & \longrightarrow & W \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{L}_t(-kx) & \longrightarrow & \mathcal{L}_t & \longrightarrow & \widetilde{W} \longrightarrow 0 \end{array}$$

Applying the functors $\mathbf{Hom}(\bullet, T')$ and $\mathbf{Hom}(\bullet, R')$ we get the following commutative diagram

$$\begin{array}{ccccccc} Y & & \mathbf{Hom}(\mathbf{i}_*\mathcal{L}_t(-kx), T') = \mathbf{Hom}(\mathcal{L}_t(-kx), R') \rightarrow \mathbf{Ext}^1(W_\bullet, R') & & & & \\ \downarrow & \nearrow & \downarrow & & \downarrow & \searrow & \uparrow \\ \mathbf{Hom}(\widetilde{\mathcal{E}}, T') & \longrightarrow & \mathbf{Ext}^1(\mathbf{i}_*\widetilde{W}, T') & = & \mathbf{Ext}^1(\widetilde{W}, R') & \longrightarrow & \mathbf{Ext}^1(W, R') \end{array}$$

and the Lemma follows. \square

Corollary 3.2.13. *We can choose a local over X_κ^{t-1} trivialization $\phi : Y \xrightarrow{\sim} H$ such that the following diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & H \\ \varepsilon \downarrow & \nearrow & \\ \mathbf{Ext}^1(W_\bullet, R') & & \end{array}$$

commutes.

Thus for every point $(E, R'_\bullet) \in Y$ we have a homomorphism $\phi_E : \mathcal{L}_t(-kx) \rightarrow R'$.

Lemma 3.2.14. *The homomorphism ϕ_E can be extended (locally over X_κ^{t-1}) to a morphism of complexes*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}_t(-kx) & \longrightarrow & \mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x) & \longrightarrow & W_\bullet \longrightarrow 0 \\ & & \phi_E \downarrow & & \psi_E \downarrow & & \parallel \\ 0 & \longrightarrow & R' & \longrightarrow & \widetilde{R}_\bullet & \longrightarrow & W_\bullet \longrightarrow 0 \end{array}$$

Proof. The claim of the Corollary 3.2.13 reformulated in terms of the derived category says that the square in the following diagram

$$\begin{array}{ccccccc}
 W_\bullet[-1] & \longrightarrow & \mathcal{L}_t(-kx) & \longrightarrow & \mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x) & \longrightarrow & W_\bullet \\
 \parallel & & \phi_E \downarrow & & & & \parallel \\
 W_\bullet[-1] & \xrightarrow{\tilde{\varepsilon}_E} & R' & \longrightarrow & \tilde{R}_\bullet & \longrightarrow & W_\bullet
 \end{array}$$

commutes. Hence it can be extended (locally) to a morphism of triangles, and the space of extensions is a torsor over $\mathbf{Hom}(W_\bullet, \tilde{R}_\bullet)$. Hence the obstruction to the local over X_κ^{t-1} extension lies in $R^1\xi_*(\mathbf{Hom}(W_\bullet, \tilde{R}_\bullet))$. But the map ξ is an affine morphism, hence the obstruction vanishes. \square

Thus we have a map $\mathbf{Hom}_Y(\tilde{R}_\bullet, R'/R'_\bullet) \xrightarrow{\psi} \mathbf{Hom}(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet)$ induced by the morphisms ψ_E . Consider the following diagram

$$\begin{array}{ccc}
 \mathbf{Hom}_Y(\tilde{R}_\bullet, R'/R'_\bullet) & \xrightarrow{\psi} & \mathbf{Hom}(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet) \\
 \downarrow & & \downarrow \\
 Y \times \mathbf{Hom}(R', R'/R'_\bullet) & \xrightarrow{\phi} & \mathbf{Hom}(\mathcal{L}_t(-kx), R'/R'_\bullet)
 \end{array}$$

Lemma 3.2.15. *The above square is cartesian.*

Proof. The Lemma 3.2.14 implies that the above square commutes, so it suffices to note that the fibers of the map $\mathbf{Hom}_Y(\tilde{R}_\bullet, R'/R'_\bullet) \rightarrow Y \times \mathbf{Hom}(R', R'/R'_\bullet)$ are equal to $\mathbf{Hom}(W_\bullet, R'/R'_\bullet)$ and that they map isomorphically to the fibers of the map $\mathbf{Hom}(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet) \rightarrow \mathbf{Hom}(\mathcal{L}_t(-kx), R'/R'_\bullet)$. \square

Corollary 3.2.16. *We have a cartesian square*

$$\begin{array}{ccc}
 X_\kappa^t & \longrightarrow & \mathbf{Hom}(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet) \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\phi} & \mathbf{Hom}(\mathcal{L}_t(-kx), R'/R'_\bullet)
 \end{array}$$

We will need the following Lemma.

Lemma 3.2.17. (i) $\mathbf{Hom}(\mathcal{L}_t(-kx), R'_\bullet) = \mathbf{Ext}^1(\mathcal{L}_t(-kx), R'_\bullet) = 0$.

(ii) *The projection $R' \rightarrow R'/R'_\bullet$ induces an isomorphism*

$$\mathbf{Hom}(\mathcal{L}_t(-kx), R') \xrightarrow{\sim} \mathbf{Hom}(\mathcal{L}_t(-kx), R'/R'_\bullet).$$

(iii) $\mathbf{Ext}^1(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R') = \mathbf{Ext}^1(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet) = 0$,

$$\dim \mathbf{Hom}(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet) = \kappa_{\leq t-1}^{\geq s}.$$

Proof. Easy. \square

Now we can finish the proof of the Proposition 3.2.9. Just note that from the definition of ϕ and from the Lemma 3.2.17 (ii) it follows that ϕ is a local (over X_κ^{t-1}) isomorphism, hence X_κ^t is locally (over X_κ^{t-1}) isomorphic to the vector bundle $\mathbf{Hom}(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet)$, which according to the Lemma 3.2.17 (iii) is $\kappa_{\leq t-1}^{\geq s}$ -dimensional.

Remark 3.2.18. It follows also that the natural map

$$\varpi_t^{-1}(E', R'_\bullet) \rightarrow \mathbf{Ext}^1(W_1, R'_1)$$

is surjective. Indeed, the map

$$\mathbf{Hom}(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet) \rightarrow \mathbf{Ext}^1(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'_\bullet)$$

is surjective by the Lemma 3.2.17 (iii). On the other hand

$$\mathbf{Ext}^1(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'_\bullet) \cong \mathbf{Ext}^1(W_\bullet, R'_\bullet)$$

by the Lemma 3.2.17 (i). So it remains to note that the projection

$$\mathbf{Ext}^1(W_\bullet, R'_\bullet) \rightarrow \mathbf{Ext}^1(W_1, R'_1)$$

is surjective and that $\varpi_t^{-1}(E', R'_\bullet) \cong \mathbf{Hom}(\mathcal{L}_t((\kappa_t^{[s,s+\bullet-1]} - k)x), R'/R'_\bullet)$.

Now we can describe the space X_κ^s .

Proposition 3.2.19. *The space X_κ^s is a pseudoaffine space of dimension*

$$\dim X_\kappa^s = \sum_{p \leq s} (s-p) \kappa_p^{\geq s} = \sum_{p \leq s \leq q \leq s+n-1} (s-p) \kappa_p^q.$$

Proof. Follows by induction from the Proposition 3.2.9. \square

Recall that our goal is to describe the space F_κ . Since we have the map $\pi_s : F_\kappa \rightarrow X_\kappa^s$ and the description of the space X_κ^s is given by the Proposition 3.2.19, it remains to describe the fiber of the map $\pi_s : F_\kappa \rightarrow X_\kappa^s$.

We will need the following Lemma.

Lemma 3.2.20. *If $E_\bullet \in F_\kappa$ then for all $p \leq q \leq s+n-1$ we have*

$$\dim \Gamma \left(\frac{E_{q+1} \cap \mathcal{E}_p}{E_q \cap \mathcal{E}_p} \right) = \kappa_{\leq p}^q.$$

Proof. Follows immediately from (4). \square

Proposition 3.2.21. *The map π_s is a pseudoaffine fibration of dimension*

$$\sum_{s \leq p < q \leq s+n-1} \kappa_{\leq p}^q = \sum_{s < p \leq q \leq s+n-1} (q-p) \kappa_p^q + \sum_{p \leq s \leq q \leq s+n-1} (q-s) \kappa_p^q.$$

Proof. Assume that $(E, R_\bullet) \in X_\kappa^s$ and let $E_\bullet \in \pi_s^{-1}(E, R_\bullet)$. Then we have a diagram

$$\begin{array}{ccccccccccc} E_s & \subset & E_{s+1} & \subset & \dots & \subset & E_{s+n-1} & \subset & E_{s+n} \\ \cap & & \cap & & & & \cap & & \cap \\ \mathcal{E}_s & \subset & \mathcal{E}_{s+1} & \subset & \dots & \subset & \mathcal{E}_{s+n-1} & \subset & \mathcal{E}_{s+n} \end{array}$$

But $E_s = E$ and $E_{s+n} = E_s(C) = E(C)$, hence every E_{s+i} is a subsheaf in $E(C) \cap \mathcal{E}_{s+i}$. In other words, we have the following diagram

$$\begin{array}{ccccccccccc} E & \subset & E_{s+1} & \subset & \dots & \subset & E_{s+n-1} & \subset & E(C) \\ \cap & & \cap & & & & \cap & & \parallel \\ E(C) \cap \mathcal{E}_s & \subset & E(C) \cap \mathcal{E}_{s+1} & \subset & \dots & \subset & E(C) \cap \mathcal{E}_{s+n-1} & \subset & E(C) \end{array}$$

Consider the following sheaves:

$$\tilde{\mathcal{E}}_i := \frac{E(C) \cap \mathcal{E}_{s+i}}{E}, \quad E'_i := \frac{E_{s+i}}{E}$$

(note that $\tilde{\mathcal{E}}_0 = R$). We have a diagram

$$\begin{array}{ccccccc} 0 & \subset & E'_1 & \subset & \dots & \subset & E'_{n-1} & \subset & \tilde{\mathcal{E}}_n \\ \cap & & \cap & & & & \cap & & \parallel \\ \tilde{\mathcal{E}}_0 & \subset & \tilde{\mathcal{E}}_1 & \subset & \dots & \subset & \tilde{\mathcal{E}}_{n-1} & \subset & \tilde{\mathcal{E}}_n \end{array}$$

and it is easy to show that for all $i \leq j$ we have

$$\frac{\tilde{\mathcal{E}}_i \cap E'_{j+1}}{\tilde{\mathcal{E}}_i \cap E'_j} = \frac{\mathcal{E}_{s+i} \cap E_{s+j+1}}{\mathcal{E}_{s+i} \cap E_{s+j}},$$

hence according to 3.2.20 we have

$$\dim \Gamma \left(\frac{\tilde{\mathcal{E}}_i \cap E'_{j+1}}{\tilde{\mathcal{E}}_i \cap E'_j} \right) = \kappa_{\leq s+i}^{s+j}. \quad (10)$$

Note also that

$$E'_i \cap \tilde{\mathcal{E}}_0 = \frac{\mathcal{E}_s \cap E_{s+i}}{E} = R_i \subset R_n = \frac{\mathcal{E}_s \cap E(1)}{E} = \tilde{\mathcal{E}}_0 \quad (11)$$

(compare with the proof of the Lemma 3.2.4). Thus, starting from E_\bullet we obtained a flag of subsheaves in the flag $\tilde{\mathcal{E}}_\bullet$ such that (10) and (11) are satisfied. On the other hand it is easy to show that if $(E'_\bullet \subset \tilde{\mathcal{E}}_\bullet)$ is such a flag, then $E_{s+i} = \mathbf{Ker}(E(C) \cap \mathcal{E}_{s+i} \rightarrow \tilde{\mathcal{E}}_i \rightarrow \tilde{\mathcal{E}}_i / E'_i)$ gives a periodic flag $E_\bullet \in \pi_s^{-1}(E, R_\bullet)$.

Thus we have to describe the space of all subflags $E'_\bullet \subset \tilde{\mathcal{E}}_\bullet$ such that (10) and (11) are satisfied. Note that all the quotients

$$\frac{\tilde{\mathcal{E}}_{i+1}}{\tilde{\mathcal{E}}_i} \subset \frac{\mathcal{E}_{s+i+1}}{\mathcal{E}_{s+i}}$$

are invertible sheaves on C , and that all the intersections $E'_i \cap \tilde{\mathcal{E}}_0$ are fixed (according to (11)), hence we can apply the standard technique of [7]. It follows that $\pi_s^{-1}(E, R_\bullet)$ is a pseudoaffine space of dimension

$$\begin{aligned} \sum_{0 \leq i < j \leq n-1} \dim \Gamma \left(\frac{\tilde{\mathcal{E}}_i \cap E'_{j+1}}{\tilde{\mathcal{E}}_i \cap E'_j} \right) &= \sum_{0 \leq i < j \leq n-1} \kappa_{\leq s+i}^{s+j} = \\ &= \sum_{s \leq p < q \leq s+n-1} \kappa_{\leq p}^q = \sum_{s < p \leq q \leq s+n-1} (q-p) \kappa_p^q + \sum_{p \leq s \leq q \leq s+n-1} (q-s) \kappa_p^q. \end{aligned}$$

□

Now the Theorem 1 follows from the Propositions 3.2.19 and 3.2.21.

3.3. Topdimensional components of K_α . Applying Theorem 1, Lemma 3.1.3 and the definition of the space C_μ^α we get the following Proposition.

Proposition 3.3.1. *Let $\mu = \{\kappa_1, \dots, \kappa_m\} \in \mathfrak{M}(\alpha)$. The stratum $K_\mu \subset K_\alpha$ is a smooth variety and*

$$\dim K_\mu = |\alpha| + (1 - \mathsf{K}(\kappa_1)) + \dots + (1 - \mathsf{K}(\kappa_m)).$$

Therefore $\dim K_\alpha = |\alpha|$ and any topdimensional irreducible component of the space K_α coincides with the closure \overline{K}_A of the stratum $K_A \subset K_\alpha$, corresponding to a Kostant partition $A \in \mathfrak{K}(\alpha)$, considered as a simple multipartition.

Proof. The first part of the Proposition is evident.

Since $K(\kappa) \geq 1$ for any Kostant partition κ it follows that for any stratum K_μ we have $\dim K_\mu \leq |\alpha|$, and equality is achieved iff $K(\kappa_1) = \dots = K(\kappa_m) = 1$, that is iff μ is a simple multipartition of α . So it remains to note that a simple multipartition is nothing but a Kostant partition. \square

4. THE SPACE \mathfrak{E}_α^i

4.1. Definition. Let $\mathfrak{E}_\alpha^i \subset K_\alpha \times K_{\alpha+i}$ denote the closed subspace formed by pairs $(E_\bullet, E'_\bullet) \in K_\alpha \times K_{\alpha+i}$ such that $E'_\bullet \subset E_\bullet$.

The embedding $\mathfrak{E}_\alpha^i \subset K_\alpha \times K_{\alpha+i}$ induces the projections

$$\mathbf{p} : \mathfrak{E}_\alpha^i \rightarrow K_\alpha \quad \text{and} \quad \mathbf{q} : \mathfrak{E}_\alpha^i \rightarrow K_{\alpha+i}.$$

If $(E_\bullet, E'_\bullet) \in \mathfrak{E}_\alpha^i$ then it is clear that $E_\bullet/E'_\bullet \cong M_i \otimes \mathcal{O}_x$ as an object of category $\mathbf{NR}_n(C)$ (here $x \in C$ and \mathcal{O}_x stands for the structure sheaf of the point x).

Let $\mathbf{r} : \mathfrak{E}_\alpha^i \rightarrow C$ denote the map sending (E_\bullet, E'_\bullet) to the point x .

Let $E'_\bullet \in K_{\alpha+i}$ and let $T'_\bullet = \mathcal{E}_\bullet/E'_\bullet$.

Lemma 4.1.1. *The fiber of the map $\mathbf{q} \times \mathbf{r} : \mathfrak{E}_\alpha^i \rightarrow K_{\alpha+i} \times C$ over the point (E'_\bullet, x) is isomorphic to the projective space*

$$\mathbb{P}(\mathbf{Hom}(M_i \otimes \mathcal{O}_x, T'_\bullet)) = \mathbb{P}(\mathbf{Hom}(\mathcal{O}_x, \mathbf{Ker}(T'_i \rightarrow T'_{i+1}))).$$

Proof. Clear. \square

Consider a piecification

$$K_{\alpha+i} \times C = \bigsqcup_{|\kappa| \leq \alpha+i} Z_{\alpha+i}^\kappa,$$

where $Z_{\alpha+i}^\kappa$ denotes the space of pairs (E'_\bullet, x) such that $\kappa(\Gamma_x(T'_\bullet)) = \kappa$.

Lemma 4.1.2. *The projection $Z_{\alpha+i}^\kappa \rightarrow C$ is a locally trivial fibration.*

Proof. Easy. \square

Lemma 4.1.3. *We have the following factorization property:*

$$Z_{\alpha+i}^\kappa \cong Z_{\alpha-\beta}^0 \times_C Z_{\beta+i}^\kappa,$$

where $\beta + i = |\kappa|$. Moreover, if $W_{\alpha+i}^\kappa = (\mathbf{q} \times \mathbf{r})^{-1}(Z_{\alpha+i}^\kappa)$ then we have a commutative diagram in which all squares are cartesian

$$\begin{array}{ccccccc} \mathfrak{E}_\beta^i & \longleftarrow & W_{\beta+i}^\kappa & \longleftarrow & W_{\alpha+i}^\kappa & \xlongequal{\quad} & \mathfrak{E}_\alpha^i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_{\beta+i} \times C & \longleftarrow & Z_{\beta+i}^\kappa & \longleftarrow & Z_{\alpha-\beta}^0 \times_C Z_{\beta+i}^\kappa & \xlongequal{\quad} & Z_{\alpha+i}^\kappa \longrightarrow K_{\alpha+i} \times C \end{array}$$

Proof. Evident. \square

The above Lemma reduces the description of the fibres of the map $\mathbf{q} \times \mathbf{r} : \mathfrak{E}_\alpha^i \rightarrow K_{\alpha+i} \times C$ to the description of the fibres of the map $W_{\beta+i}^\kappa \rightarrow Z_{\beta+i}^\kappa$, where $|\kappa| = \beta + i$. It is clear that $Z_{\beta+i}^\kappa \rightarrow C$ is a locally trivial F_κ -fibration. We fix some point $x \in C$ and consider F_κ as the fiber of $Z_{\beta+i}^\kappa$ over the point x . Let $\tilde{F}_\kappa = (\mathbf{q} \times \mathbf{r})^{-1}(F_\kappa) \subset W_{\beta+i}^\kappa$.

Finally, let $Z_{\beta+i}^\kappa(r) \subset Z_{\beta+i}^\kappa$ be the subspace of points with the dimension of the fiber of the map $W_{\beta+i}^\kappa \rightarrow Z_{\beta+i}^\kappa$ greater than or equal to r , and let $F_\kappa(r) \subset F_\kappa$ be

the subspace of points with the dimension of the fiber of the map $\tilde{F}_\kappa \rightarrow F_\kappa$ greater than or equal to r .

Lemma 4.1.4. *The map $Z_{\beta+i}^\kappa(r) \rightarrow C$ is a locally trivial $F_\kappa(r)$ -fibration.*

Proof. Clear. \square

Proposition 4.1.5. *The subspace $F_\kappa(r) \subset F_\kappa$ is empty if $K(\kappa) < r+1$ and has codimension at least r if $K(\kappa) \geq r+1$.*

The proof of the Proposition will be given in the next subsection. Now we will deduce from it the following Theorem.

Theorem 2. *The map $\mathbf{q} : \mathfrak{E}_\alpha^i \rightarrow K_{\alpha+i}$ is semismall.*

Proof. It follows from the Proposition 4.1.5 and from the Lemma 4.1.4 that the subspace $Z_{\beta+i}^\kappa(r) \subset Z_{\beta+i}^\kappa$ is empty if $K(\kappa) < r+1$ and has codimension at least r if $K(\kappa) \geq r+1$. From the Lemma 4.1.2 and 4.1.3 it follows that the map $Z_{\alpha+i}^\kappa \rightarrow Z_{\beta+i}^\kappa$ is a locally trivial fibration, hence the subspace $Z_{\alpha+i}^\kappa(r) \subset Z_{\alpha+i}^\kappa$ is empty if $K(\kappa) < r+1$ and has codimension at least r if $K(\kappa) \geq r+1$.

Now let $X(r) \subset K_{\alpha+i} \times C$ be the subspace of points with the dimension of the fiber of the map $\mathfrak{E}_\alpha^i \rightarrow K_{\alpha+i} \times C$ greater than or equal to r . Then

$$X(r) = \bigsqcup_{|\kappa| \leq \alpha+i} (X(r) \cap Z_{\alpha+i}^\kappa) = \bigsqcup_{|\kappa| \leq \alpha+i} Z_{\alpha+i}^\kappa(r).$$

Hence

$$\text{codim}_{K_{\alpha+i} \times C} X(r) = \min \text{codim}_{K_{\alpha+i} \times C} Z_{\alpha+i}^\kappa(r).$$

But

$$\text{codim}_{K_{\alpha+i} \times C} Z_{\alpha+i}^\kappa(r) = \text{codim}_{Z_{\alpha+i}^\kappa} Z_{\alpha+i}^\kappa(r) + \text{codim}_{K_{\alpha+i} \times C} Z_{\alpha+i}^\kappa,$$

and if $Z_{\alpha+i}^\kappa(r)$ is not empty then

$$\text{codim}_{K_{\alpha+i} \times C} Z_{\alpha+i}^\kappa = K(\kappa) \geq r+1 \quad \text{and} \quad \text{codim}_{Z_{\alpha+i}^\kappa} Z_{\alpha+i}^\kappa(r) \geq r,$$

hence $\text{codim } X(r) \geq 2r+1$.

So it remains to note that the fiber of the map $\mathbf{q} : \mathfrak{E}_\alpha^i \rightarrow K_{\alpha+i}$ over the point E'_\bullet is equal to the disjoint union of the fibers of the map $\mathbf{q} \times \mathbf{r}$ over the finite number of points (E'_\bullet, x) , hence we have $K_{\alpha+i}(r) \subset p_1(X(r))$, where $K_{\alpha+i}(r)$ is the subspace of points with the dimension of the fiber of the map $\mathfrak{E}_\alpha^i \rightarrow K_{\alpha+i}$ greater than or equal to r , and $p_1 : K_{\alpha+i} \times C \rightarrow K_{\alpha+i}$ is the projection. Hence, $\text{codim } K_{\alpha+i}(r) \geq \text{codim } X(r) - 1 \geq 2r$. \square

4.2. Space $F_\kappa(r)$. This subsection is devoted to the proof of the Proposition 4.1.5. Here we use the notation of the section 3.2. Recall that there we fixed an arbitrary integer s . Let us take $s = i \bmod n$. Then we have

$$\mathbf{Ker}(T_i \rightarrow T_{i+1}) = R_1$$

according to the definition of R_1 . Hence the subspace $F_\kappa(r) \subset F_\kappa$ is given by the condition

$$\dim \mathbf{Hom}(\mathcal{O}_x, R_1) \geq r+1.$$

Recall also that we have a locally trivial fibration $F_\kappa \rightarrow X_\kappa^s$ and a sequence of locally trivial fibrations

$$X_\kappa^s \xrightarrow{\varpi_s} X_\kappa^{s-1} \xrightarrow{\varpi_{s-1}} \dots$$

Let $X_\kappa^t(r) \subset X_\kappa^t$ be the subspace of points (E, R_\bullet) such that $\dim \mathbf{Hom}(\mathcal{O}_x, R_1) = r+1$. Let $N_t = \#\{p \leq t \mid \kappa_p^s > 0\}$. Then it is clear that $N_s \leq K(\kappa)$. Hence it suffices to prove the following Proposition.

Proposition 4.2.1. *The subspace $X_\kappa^t(r) \subset X_\kappa^t$ is empty if $N_t < r+1$ and has codimension at least r if $N_t \geq r+1$.*

Proof. We apply the induction in t .

The base of the induction is evident: if $t \ll s$ we have $N_t = 0$ and $X_\kappa^t(r)$ is empty for any r (because $R_1 = 0$).

So assume that the induction hypothesis for $t-1$ is true. We have two cases: $\kappa_t^s = 0$ and $\kappa_t^s > 0$.

If $\kappa_t^s = 0$ then it is evident that $N_t = N_{t-1}$ and that $R_1 = R'_1$, hence $X_\kappa^t(r) = \varpi_t^{-1}(X_\kappa^{t-1}(r))$ and the induction hypothesis for $t-1$ and t are equivalent.

If $\kappa_t^s > 0$ then $N_t = N_{t-1} + 1$ and we have the following exact sequence

$$0 \rightarrow R'_1 \rightarrow R_1 \rightarrow W_1 \rightarrow 0 \quad (12)$$

and $W_1 \cong \mathcal{L}_t/\mathcal{L}_t(-\kappa_t^s x)$. It follows that

$$X_\kappa^t(r) \subset \varpi_t^{-1}(X_\kappa^{t-1}(r-1)) \bigcup \varpi_t^{-1}(X_\kappa^{t-1}(r)).$$

Hence it suffices to check that

$$\text{codim}_{\varpi_t^{-1}(X_\kappa^{t-1}(r-1))}(X_\kappa^t(r) \cap \varpi_t^{-1}(X_\kappa^{t-1}(r-1))) \geq 1.$$

The latter condition will be satisfied if we prove that for any point of $X_\kappa^{t-1}(r-1)$ a generic point of the fiber of the map ϖ_t over this point belongs to $X_\kappa^t(r-1)$.

But this is true, because we have $\dim \mathbf{Hom}(\mathcal{O}_x, R_1) = \dim \mathbf{Hom}(\mathcal{O}_x, R'_1)$ for a generic extension (12) and according to the Remark 3.2.18 all types of extensions (12) are realized in the fiber of the map ϖ_t over any point of X_κ^{t-1} . \square

4.3. Irreducible components of \mathfrak{E}_α^i . We will need a description of irreducible components of \mathfrak{E}_α^i which are dominant over topdimensional components of the spaces K_α and $K_{\alpha+i}$.

We begin with the case of $K_{\alpha+i}$. Let $A' \in \mathfrak{K}(\alpha+i)$ be a Kostant partition. We will consider A' as a simple multipartition. Let $K_{A'}$ be the corresponding stratum of $K_{\alpha+i}$ and let $\overline{K}_{A'}$ be the corresponding $(|\alpha|+1)$ -dimensional irreducible component.

Proposition 4.3.1. *If $A' = \{\{\theta'_1, \dots, \theta'_m\}\}$ then the components of \mathfrak{E}_α^i dominant over $\overline{K}_{A'}$ are in one-to-one correspondence with elements θ'_r of the partition A' ending at i . The projection $\mathbf{p} : \mathfrak{E}_\alpha^i \rightarrow K_\alpha$ sends the generic point of the component of \mathfrak{E}_α^i , corresponding to the element θ'_r of the partition A' , to the generic point of the component \overline{K}_A of K_α with $A = \{\{\theta'_1, \dots, \theta'_r/i, \dots, \theta'_m\}\}$.*

Proof. Let E'_\bullet be a generic point of $K_{A'}$ and let $\sigma(E'_\bullet) = \theta'_1 x_1 + \dots + \theta'_m x_m$. It follows evidently from the Lemma 4.1.1 that the fiber of the map $\mathbf{q} \times \mathbf{r}$ over the point (E'_\bullet, x) is a point, if $x = x_r$ and $\theta'_r \in \mathbf{E}_i$ (that is if M_i is a subrepresentation in $M_{\theta'_r} = \Gamma((T'_x)_\bullet)$), and is empty otherwise. Hence, the fiber of \mathbf{q} over the point E'_\bullet is finite and points in the fiber are in a bijection with elements of the partition A' ending at i .

Let us take a point $(E_\bullet \supset E'_\bullet) \in \mathfrak{E}_\alpha^i$ in the fiber of \mathbf{q} over a generic point $E'_\bullet \in K_{A'} \subset \overline{K}_{A'}$, corresponding to an element θ'_r of the Kostant partition A' . Then it is clear that for all $x \neq x_r$ we have $(T_x)_\bullet = (T'_x)_\bullet$ and for $x = x_r$ we have

$$0 \rightarrow M_i \otimes \mathcal{O}_x \rightarrow (T'_x)_\bullet \rightarrow (T_x)_\bullet \rightarrow 0,$$

hence $\Gamma((T_x)_\bullet) = \Gamma((T'_x)_\bullet)/M_i = M_{\theta'_r}/M_i = M_{\theta'_r/i}$, hence $E_\bullet \in K_A$ with $A = \{\{\theta'_1, \dots, \theta'_r/i, \dots, \theta'_m\}\}$ and the Proposition follows. \square

Let $\mathfrak{E}_{A'}(\theta')$ denote the component of \mathfrak{E}_α^i dominant over $\overline{K}_{A'}$, corresponding to the element θ' of the Kostant partition A' .

Lemma 4.3.2. *The map $\mathbf{q} : \mathfrak{E}_{A'}(\theta') \rightarrow \overline{K}_{A'}$ is a generically finite map of the degree equal to $m(\theta', A')$.*

Proof. Evident. \square

The description of the components of \mathfrak{E}_α^i which are dominant over $|\alpha|$ -dimensional components of K_α is more difficult. Let $A \in \mathfrak{K}(\alpha)$ be a Kostant partition. We consider A as a simple multipartition. Let K_A be the corresponding stratum of K_α and let \overline{K}_A be the corresponding irreducible component.

Let $E_\bullet \in K_A$ be a generic point.

Lemma 4.3.3. *The fiber of the map $(\mathbf{p} \times \mathbf{r}) : \mathfrak{E}_\alpha^i \rightarrow K_\alpha \times C$ over the point (E_\bullet, x) is isomorphic to the projective space*

$$\mathbb{P}(\mathbf{Hom}(E_\bullet, M_i \otimes \mathcal{O}_x)) = \mathbb{P}(\mathbf{Hom}(E_i/E_{i-1}, \mathcal{O}_x)).$$

Proof. Easy. \square

Assume that $A = \{\{\theta_1, \dots, \theta_m\}\}$ and $\sigma(E_\bullet) = \{\{\theta_1\}\}x_1 + \dots + \{\{\theta_m\}\}x_m$.

Lemma 4.3.4. *If $x = x_r$ and the raiz $\theta_r \in E_{i-1}$ then we have*

$$\mathbb{P}(\mathbf{Hom}(E_i/E_{i-1}, \mathcal{O}_x)) = \mathbb{P}^1$$

and otherwise $\mathbb{P}(\mathbf{Hom}(E_i/E_{i-1}, \mathcal{O}_x))$ is a point.

Proof. From the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_{i-1} & \longrightarrow & E_i & \longrightarrow & E_i/E_{i-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}_{i-1} & \longrightarrow & \mathcal{E}_i & \longrightarrow & \mathbf{i}_* \mathcal{L}_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & T_{i-1} & \longrightarrow & T_i & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

it follows that E_i/E_{i-1} is a direct sum of a line bundle on the curve C and of the torsion sheaf $\mathbf{Ker}(T_{i-1} \rightarrow T_i)$. Hence

$$\dim \mathbf{Hom}(E_i/E_{i-1}, \mathcal{O}_x) = 1 + \dim \mathbf{Hom}(\mathbf{Ker}(T_{i-1} \rightarrow T_i), \mathcal{O}_x).$$

So it remains to note that the sheaf $\mathbf{Ker}(T_{i-1} \rightarrow T_i)$ has a nontrivial component at the point x only if $x = x_r$ and $\theta_r \in E_{i-1}$, and that in this case the component is isomorphic to \mathcal{O}_x . \square

Corollary 4.3.5. *The fiber $p^{-1}(E_\bullet)$ is a reducible curve*

$$\mathbf{C} = C_0 \cup \left(\bigcup_{\{r \mid \theta_r \in \mathbf{E}_{i-1}\}} C_r \right),$$

(see the Figure 1). All vertical components of the curve \mathbf{C} are genus 0 curves. The horizontal component C_0 of \mathbf{C} maps isomorphically to C under the map \mathbf{r} (hence $g(C_0) = g(C)$), while the vertical components C_r are contracted by \mathbf{r} to the points $x_r \in C$.

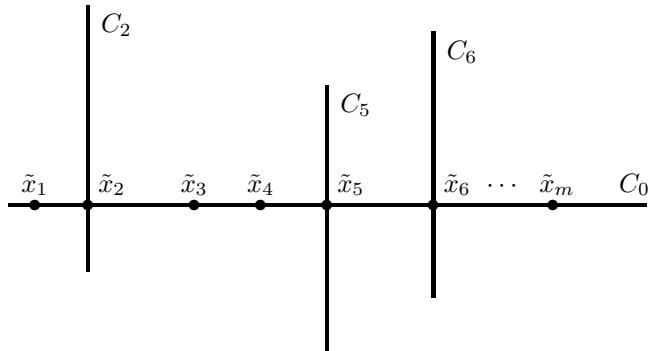


FIGURE 1. The curve \mathbf{C}

Let $\tilde{x}_r = \mathbf{r}^{-1}(x_r) \cap C_0$ be the preimages of the points x_r ($r = 1, \dots, m$) on the component C_0 of \mathbf{C} . Thus if $\theta_r \in E_{i-1}$ then \tilde{x}_r is the point of intersection of the components C_0 and C_r of \mathbf{C} .

Lemma 4.3.6. *Let $\tilde{x} \in \mathbf{C}$. Then we have*

$$\mathbf{q}(\tilde{x}) \in \begin{cases} K_{\{\{\theta_1, \dots, \theta_r, \star i, \dots, \theta_m\}\}} \subset K_{\alpha+i}, & \text{if } \tilde{x} \in C_r - \{\tilde{x}_r\}, \\ K_{\{\{\theta_1, \dots, \theta_r, \dots, \theta_m, i\}\}} \subset K_{\alpha+i}, & \text{if } \tilde{x} \in C_0 - \{\tilde{x}_1, \dots, \tilde{x}_m\}, \\ K_{\{\{\theta_1, \dots, \{\theta_r, i\}, \dots, \theta_m\}\}} \subset K_{\alpha+i}, & \text{if } \tilde{x} = \tilde{x}_r. \end{cases}$$

Proof. Easy.

Lemma 4.3.7. Consider the stratum $K_{\{\{\theta_1, \dots, \{\{\theta_r, i\}\}, \dots, \theta_m\}\}}$ of $K_{\alpha+i}$. Then it lies in the component $\overline{K}_{\{\{\theta_1, \dots, \theta_r, \dots, \theta_m, i\}\}}$ of $K_{\alpha+i}$. Other $(|\alpha|+1)$ -dimensional components of $K_{\alpha+i}$, containing this stratum are listed below:

This list is complete.

Proof. Evident.

Given a Kostant partition $A' \in \mathfrak{K}(\alpha + i)$ consider the intersection

$$\mathfrak{E}_A^{A'} = \mathfrak{E}_\alpha^i \cap p^{-1}(\overline{K}_A) \cap q^{-1}(\overline{K}_{A'}).$$

We are interested in irreducible components of $\mathfrak{E}_A^{A'}$ which are dominant over \overline{K}_A .

Proposition 4.3.8. *If $A' = \{\{\theta_1, \dots, \theta_r, \dots, \theta_m, i\}\}$ then $\mathfrak{E}_A^{A'}$ has only one irreducible component, dominant over \overline{K}_A . This component is a generically C -fibration over \overline{K}_A . Its fiber over a generic point $E_\bullet \in K_A \subset \overline{K}_A$ is equal to the component C_0 of the curve $\mathbf{C} = \mathbf{p}^{-1}(E_\bullet)$.*

Proof. Easy. □

Proposition 4.3.9. *If $\theta_r \in E_{i-1}$ and $A' = \{\{\theta_1, \dots, \theta_r \star i, \dots, \theta_m\}\}$ then $\mathfrak{E}_A^{A'}$ has only one irreducible component, dominant over \overline{K}_A . This component is a generically $(\underbrace{\mathbb{P}^1 \sqcup \dots \sqcup \mathbb{P}^1}_{m(\theta_r, A) \text{ times}})$ -fibration over \overline{K}_A . Its fiber over a generic point $E_\bullet \in K_A \subset \overline{K}_A$ is equal to the disjoint union of all components $C_{r'}$ of the curve $\mathbf{C} = \mathbf{p}^{-1}(E_\bullet)$ with $\theta_{r'} = \theta_r$.*

Proof. Easy. □

Proposition 4.3.10. *If $\theta_r \in B_{i+1}$ and $A' = \{\{\theta_1, \dots, i \star \theta_r, \dots, \theta_m\}\}$ then $\mathfrak{E}_A^{A'}$ has only one irreducible component, dominant over \overline{K}_A . This component is a generically finite $m(\theta_r, A)$ -fold covering of \overline{K}_A . Its fiber over a generic point $E_\bullet \in K_A \subset \overline{K}_A$ is equal to the set all points $\tilde{x}_{r'}$ of the curve $\mathbf{C} = \mathbf{p}^{-1}(E_\bullet)$ such that $\theta_{r'} = \theta_r$.*

Proof. Easy. □

5. $\widehat{\mathfrak{sl}}_n$ -MODULE STRUCTURE

5.1. Preliminaries. We want to introduce a $\widehat{\mathfrak{sl}}_n$ -module structure on the vector space

$$\mathbf{M} = \bigoplus_{\alpha \in \mathbb{N}[I]} H^0(K_\alpha, \mathbb{Q}).$$

This space is naturally Y -graded. Recall that the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ is given by the generators e_i, f_i, h_i , ($i \in I$) (Chevalley generators) which satisfy the following set of relations (Serre relations)

$$\begin{aligned} \mathbf{ad}(e_i)^{(1-a_{ij})} e_j &= 0, & (i \neq j \in I) \\ \mathbf{ad}(f_i)^{(1-a_{ij})} f_j &= 0, & (i \neq j \in I) \\ [e_i, f_j] &= \delta_{ij} h_i, & (i, j \in I) \\ [h_i, e_j] &= a_{ij} e_j, & (i, j \in I) \\ [h_i, f_j] &= -a_{ij} f_j, & (i, j \in I) \\ [h_i, h_j] &= 0, & (i, j \in I) \end{aligned}$$

Recall also, that $\mathbf{c} = \sum_{i \in I} h_i$ is the central element of $\widehat{\mathfrak{sl}}_n$.

5.2. Definition of e_i . We choose the natural \mathbb{Q} -basis in the spaces $H^0(K_\alpha, \mathbb{Q})$, formed by the fundamental classes of topdimensional irreducible components of K_α . Let $v_A = [\overline{K}_A] \in H^0(K_\alpha, \mathbb{Q})$ be the fundamental class of the component \overline{K}_A , corresponding to a Kostant partition $A \in \mathfrak{K}(\alpha)$.

We will begin with the definition of the action of the Chevalley generators e_i , f_i and h_i and after that we will check that the Serre relations between them are satisfied.

Definition 5.2.1. The operator $e_i : H^0(K_\alpha, \mathbb{Q}) \rightarrow H^0(K_{\alpha+i}, \mathbb{Q})$ is given by the correspondence $\mathfrak{E}_\alpha^i \subset K_\alpha \times K_{\alpha+i}$.

Let $\varepsilon_i(A, A')$ denote the matrix coefficient of the operator e_i with respect to our bases. Thus for all $A \in \mathfrak{K}(\alpha)$ we have

$$e_i(v_A) = \sum_{A' \in \mathfrak{K}(\alpha+i)} \varepsilon_i(A, A') v_{A'}. \quad (13)$$

Proposition 5.2.2. Let $A' = \{\{\theta'_1, \dots, \theta'_m\}\}$. We have

$$\varepsilon_i(A, A') = \begin{cases} m(\theta_r, A'), & \text{if } \theta'_r \in \mathfrak{E}_i \text{ and } A = \{\{\theta'_1, \dots, \theta'_r/i, \dots, \theta'_m\}\} \\ 0, & \text{otherwise} \end{cases}$$

Proof. Follows from the Proposition 4.3.1 and from the Lemma 4.3.2. \square

5.3. Definition of h_i .

Definition 5.3.1. We define the operator $h_i : H^0(K_\alpha, \mathbb{Q}) \rightarrow H^0(K_\alpha, \mathbb{Q})$ as a scalar multiplication by $\langle i', (2-2g)\rho + \alpha_0 + \alpha \rangle$.

5.4. Definition of f_i . The definition of the operator f_i is rather more complicated. The reason is that the dimension of \mathfrak{E}_α^i is equal to $|\alpha| + 1$, so it doesn't define an operator $H^0(K_{\alpha+i}, \mathbb{Q}) \rightarrow H^0(K_\alpha, \mathbb{Q})$. Thus, as in the case of excess intersection we should introduce certain second cohomology classes $\xi_\alpha^i \in H^2(\mathfrak{E}_\alpha^i)$.

This can be done as follows. Consider the space $\mathfrak{E}_\alpha^i \times S$ and let $E_\bullet \subset \mathcal{O}_{\mathfrak{E}_\alpha^i} \boxtimes \mathcal{E}_\bullet$ (resp. $E'_\bullet \subset \mathcal{O}_{\mathfrak{E}_\alpha^i} \boxtimes \mathcal{E}_\bullet$) denote the universal degree- α (resp. degree- $(\alpha + i)$) periodic subflag on $\mathfrak{E}_\alpha^i \times S$. We have the universal embedding $E'_\bullet \subset E_\bullet$ which gives the exact sequence

$$0 \rightarrow E'_i \rightarrow E_i \rightarrow (\mathbf{id} \times \mathbf{i})_* \Delta_*^i L_\alpha^i \rightarrow 0$$

where L_α^i is a line bundle on \mathfrak{E}_α^i and the embedding Δ^i is defined from the following cartesian square

$$\begin{array}{ccccc} \mathfrak{E}_\alpha^i \times S & \xleftarrow{\mathbf{id} \times \mathbf{i}} & \mathfrak{E}_\alpha^i \times C & \xleftarrow{\Delta^i} & \mathfrak{E}_\alpha^i \\ \mathbf{r} \times \mathbf{id} \downarrow & & \mathbf{r} \times \mathbf{id} \downarrow & & \mathbf{r} \downarrow \\ C \times S & \xleftarrow{\mathbf{id} \times \mathbf{i}} & C \times C & \xleftarrow{\Delta} & C \end{array}$$

Consider also the following commutative diagram of sheaves on $K_\alpha \times S$

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_{i+1} \cap \mathcal{E}_i & \longrightarrow & E_{i+1} & \longrightarrow & E_{i+1}/(E_{i+1} \cap \mathcal{E}_i) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}_i & \longrightarrow & \mathcal{E}_{i+1} & \longrightarrow & (\mathbf{id} \times \mathbf{i})_* \mathcal{L}_{i+1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T'_i & \longrightarrow & T_{i+1} & \longrightarrow & T_{i+1}/T'_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \quad (14)$$

where $T'_i = \mathcal{E}_i / (E_{i+1} \cap \mathcal{E}_i) = \mathbf{Im}(T_i \rightarrow T_{i+1})$. It follows that $E_{i+1} / (E_{i+1} \cap \mathcal{E}_i)$ is a subsheaf in $(\mathbf{id} \times \mathbf{i})_* \mathcal{L}_{i+1}$, hence

$$E_{i+1} / (E_{i+1} \cap \mathcal{E}_i) \cong (\mathbf{id} \times \mathbf{i})_* \mathcal{F}_\alpha^i$$

where \mathcal{F}_α^i is a torsion free rank 1 sheaf on $K_\alpha \times C$.

Definition 5.4.1. Let

$$\xi_\alpha^i = c_1((\Delta^i)^* (\mathbf{p} \times \mathbf{id})^* \mathcal{F}_\alpha^i) - c_1(L_\alpha^i) + c_1(\mathbf{r}^* \mathcal{T}_C) \in H^2(\mathfrak{E}_\alpha^i, \mathbb{Z}),$$

where \mathcal{T}_C stands for the tangent bundle of the curve C .

It is clear that first and second summands here depend not only on $i \bmod n$ but on i itself also. However, it will be seen from the proof of the Proposition 5.4.3 that the class ξ_α^i depend only on $i \bmod n$.

Definition 5.4.2. Let $A \in \mathfrak{K}(\alpha)$, $A' \in \mathfrak{K}(\alpha + i)$ be a pair of Kostant partitions. We define the operator $f_A^{A'} : H^0(\overline{K}_{A'}) \rightarrow H^0(\overline{K}_A)$ as follows. If the intersection $\mathfrak{E}_A^{A'}$ is $(|\alpha| + 1)$ -dimensional, then $f_A^{A'}$ is given by the class $(\xi_\alpha^i)|_{\mathfrak{E}_A^{A'}}$ (that is, $f_A^{A'}$ equals the integral of ξ_α^i over a generic fiber of $\mathfrak{E}_A^{A'}$ over K_A). If the intersection $\mathfrak{E}_A^{A'}$ is $|\alpha|$ -dimensional, then $f_A^{A'}$ is given by $\mathfrak{E}_A^{A'}$ (that is, $f_A^{A'}$ equals the cardinality of a generic fiber of $\mathfrak{E}_A^{A'}$ over K_A). Finally, we define the operator $f_i : H^0(K_{\alpha+i}, \mathbb{Q}) \rightarrow H^0(K_\alpha, \mathbb{Q})$ as the sum of operators $f_A^{A'}$ for all pairs (A, A') :

$$f_i = \sum_{A \in \mathfrak{K}(\alpha), A' \in \mathfrak{K}(\alpha+i)} f_A^{A'}.$$

Let $\phi_i(A', A)$ denote the matrix coefficient of the operator f_i with respect to our bases. Thus for all $A' \in \mathfrak{K}(\alpha + i)$ we have

$$f_i(v_{A'}) = \sum_{A \in \mathfrak{K}(\alpha)} \phi_i(A', A) v_A. \quad (15)$$

Proposition 5.4.3. Let $A = \{\{\theta_1, \dots, \theta_m\}\}$. We have

$$\phi_i(A', A) = \begin{cases} M(i, A), & \text{if } A' = \{\{\theta_1, \dots, \theta_m, i\}\}, \\ -m(\theta_r, A), & \text{if } \theta_r \in \mathsf{E}_{i-1} \text{ and } A' = \{\{\theta_1, \dots, \theta_r \star i, \dots, \theta_m\}\} \\ m(\theta_r, A), & \text{if } \theta_r \in \mathsf{B}_{i+1} \text{ and } A' = \{\{\theta_1, \dots, i \star \theta_r, \dots, \theta_m\}\} \\ 0, & \text{otherwise} \end{cases}$$

where

$$M(i, A) = \langle i', (2 - 2g)\rho + \alpha_0 \rangle + \sum_{\theta \in \mathsf{B}_{i+1}} (m(i \star \theta, A) - m(\theta, A)).$$

Proof. The third case follows immediately from the Proposition 4.3.10 and from the definition of the operator $f_A^{A'}$. In order to check the first two cases we should compute the restriction of the class ξ_α^i to the components C_0 and C_r of the curve $\mathbf{C} = \mathbf{p}^{-1}(E_\bullet)$, where E_\bullet is a generic point of the stratum $K_A \subset K_\alpha$.

Recall that ξ_α^i is defined as a sum of three summands. We begin with the computation of $((\Delta^i)^* (\mathbf{p} \times \mathbf{id})^* \mathcal{F}_\alpha^i)|_{\mathbf{C}}$. Note that $(\mathbf{p} \times \mathbf{id}) \cdot \Delta^i = \mathbf{p} \times \mathbf{r}$, hence

$(\Delta^i)^*(\mathbf{p} \times \mathbf{id})^* \mathcal{F}_\alpha^i = (\mathbf{p} \times \mathbf{r})^* \mathcal{F}_\alpha^i$. Since $\mathbf{C} = \mathbf{p}^{-1}(E_\bullet)$, we have the following cartesian square

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathfrak{E}_\alpha^i \\ \mathbf{r} \downarrow & & \mathbf{p} \times \mathbf{r} \downarrow \\ \{E_\bullet\} \times C & \longrightarrow & K_\alpha \times C \end{array}$$

hence $((\mathbf{p} \times \mathbf{r})^* \mathcal{F}_\alpha^i)|_{\mathbf{C}} = \mathbf{r}^*(\mathcal{F}_\alpha^i|_{\{E_\bullet\} \times C})$. It follows that this bundle is trivial on vertical components of \mathbf{C} over C , that is

$$c_1(((\Delta^i)^*(\mathbf{p} \times \mathbf{id})^* \mathcal{F}_\alpha^i)|_{C_r}) = 0.$$

At the same time its restriction to C_0 is isomorphic to $\mathcal{F}_\alpha^i|_{\{E_\bullet\} \times C}$, which according to the definition of \mathcal{F}_α^i is equal to $E_{i+1}/(E_{i+1} \cap \mathcal{E}_i)$. The diagram (14) implies that $E_{i+1}/(E_{i+1} \cap \mathcal{E}_i) \cong \mathbf{i}_*(\mathcal{L}_{i+1}(-\dim \Gamma(T_{i+1}/T'_i)))$. So, finally we get

$$\begin{aligned} c_1(((\Delta^i)^*(\mathbf{p} \times \mathbf{id})^* \mathcal{F}_\alpha^i)|_{C_0}) &= \deg \mathcal{L}_{i+1}(-\dim \Gamma(T_{i+1}/T'_i)) = \\ &= \deg \mathcal{L}_{i+1} - \dim \Gamma(T_{i+1}/T'_i) = \deg \mathcal{L}_{i+1} - \sum_{\theta \in \mathbf{B}_{i+1}} m(\theta, A). \end{aligned}$$

The second summand can be computed as follows. First consider a vertical component C_r . Note that C_r is by definition isomorphic to the projective line $\mathbb{P}(H_r)$, where $H_r = \mathbf{Hom}(E_\bullet, M_i \otimes \mathcal{O}_{x_r}) = \mathbf{Hom}(E_i/E_{i-1}, \mathcal{O}_{x_r})$. Consider the natural morphism of coherent sheaves on $C_r \times S$

$$\mathcal{O}_{C_r} \boxtimes E_i \rightarrow \mathcal{O}_{C_r} \boxtimes E_i/E_{i-1} \rightarrow \mathcal{O}_{C_r} \boxtimes (H_r^* \otimes \mathcal{O}_{x_r}) \rightarrow \mathcal{O}_{C_r}(1) \boxtimes \mathcal{O}_{x_r}$$

It is clear that the kernel of this morphism is nothing but the restriction of the sheaf E'_i from $\mathfrak{E}_\alpha^i \times S$ to $C_r \times S$. Hence $L_{\alpha|C_r}^i \cong \mathcal{O}_{C_r}(1)$ and

$$c_1(L_{\alpha|C_r}^i) = 1.$$

Now we will treat the component C_0 . Consider the following morphism of coherent sheaves on $C_0 \times S$

$$\begin{aligned} \mathcal{O}_{C_0} \boxtimes E_i &\rightarrow \mathcal{O}_{C_0} \boxtimes E_i/E_{i-1} \rightarrow \mathcal{O}_{C_0} \boxtimes E_i/(E_i \cap \mathcal{E}_{i-1}) \rightarrow \\ &\rightarrow \mathcal{O}_{C_0} \boxtimes \mathbf{i}_*\mathcal{L}_i(-\dim \Gamma(T_i/T'_{i-1})) \rightarrow (\mathbf{id} \times \mathbf{i})_*\mathcal{L}_i(-\dim \Gamma(T_i/T'_{i-1})) \end{aligned}$$

It is clear that the kernel of this morphism is just the restriction of the sheaf E'_i from $\mathfrak{E}_\alpha^i \times S$ to $C_0 \times S$. Hence $L_{\alpha|C_0}^i \cong \mathcal{L}_i(-\dim \Gamma(T_i/T'_{i-1}))$ and

$$c_1(L_{\alpha|C_0}^i) = \deg \mathcal{L}_i - \dim \Gamma(T_i/T'_{i-1}) = \deg \mathcal{L}_i - \sum_{\theta \in \mathbf{B}_i} m(\theta, A).$$

Finally, we should compute the third summand. But it is evident that the bundle $\mathbf{r}^*\mathcal{T}_C$ is trivial on the vertical components of the curve \mathbf{C} and is isomorphic to \mathcal{T}_{C_0} on the component C_0 . So summing up all contributions we obtain

$$\begin{aligned} \xi_{\alpha|C_0}^i &= \deg \mathcal{L}_{i+1} - \sum_{\theta \in \mathbf{B}_{i+1}} m(\theta, A) - (\deg \mathcal{L}_i - \sum_{\theta \in \mathbf{B}_i} m(\theta, A)) + 2 - 2g = \\ &= 2 - 2g + \langle i', \alpha_0 \rangle + (\sum_{\theta \in \mathbf{B}_i} m(\theta, A)) - (\sum_{\theta \in \mathbf{B}_{i+1}} m(\theta, A)) = M(i, A) \end{aligned}$$

in the case of the component C_0 and

$$\xi_{\alpha|C_r}^i = 0 - 1 + 0 = -1$$

in the case of the component C_r . So it remains to recall that according to the Proposition 4.3.9 the number of vertical components, contributing to the coefficient $\phi_i(A', A)$ is equal to $m(\theta_r, A)$. \square

We finish this section with the following.

Theorem 3. *The operators e_i, h_i, f_i ($i \in I$) provide the vector space \mathbf{M} with a structure of $\widehat{\mathfrak{sl}}_n$ -module with the lowest weight $(2-2g)\rho + \alpha_0$ and the central charge $c_0 = (2-2g)n + d$.*

6. SERRE RELATIONS

This section is devoted to the proof of the Theorem 3.

6.1. Representation in differential operators. Let $\mathbf{N} = \mathbb{Q}[x_\theta]$ denote a Y -graded vector space of polynomials in variables x_θ , $\theta \in R^+$, with the inverse grading

$$\deg x^\theta = -\dim \theta$$

Given a Kostant partition $A = \{\{\theta_1, \dots, \theta_m\}\} \in \mathfrak{K}(\alpha)$ let x^A denote the following monomial

$$x^A = x_{\theta_1} \cdots x_{\theta_m} \in \mathbf{N}_{-\alpha}.$$

It is clear that monomials x^A with $A \in \mathfrak{K}(\alpha)$ form a basis of $\mathbf{N}_{-\alpha}$. We define the pairing $\mathbf{N} \otimes \mathbf{M} \rightarrow \mathbb{Q}$ as follows

$$\langle x^A, v_{A'} \rangle = \delta_{A, A'}.$$

This gives an isomorphism $\mathbf{N} \cong \mathbf{M}^*$. Let

$$e_i^T : \mathbf{N}_{-\alpha-i} \rightarrow \mathbf{N}_{-\alpha}, \quad h_i^T : \mathbf{N}_{-\alpha} \rightarrow \mathbf{N}_{-\alpha}, \quad f_i^T : \mathbf{N}_{-\alpha} \rightarrow \mathbf{N}_{-\alpha-i},$$

denote the adjoint operators of the operators e_i , h_i and f_i , defined in the previous section.

Lemma 6.1.1. *We have*

$$e_i^T(x^{A'}) = \sum_{A \in \mathfrak{K}(\alpha)} \varepsilon_i(A, A') x^A, \quad f_i^T(x^A) = \sum_{A' \in \mathfrak{K}(\alpha+i)} \phi_i(A', A) x^{A'},$$

where $\varepsilon_i(A, A')$ and $\phi_i(A', A)$ are given by the Propositions 5.2.2 and 5.4.3, and $h_i^T|_{\mathbf{N}_{-\alpha}}$ is the scalar multiplication by $\langle i', (2-2g)\rho + \alpha_0 + \alpha \rangle$.

Proof. Evident. \square

The following Proposition shows that the operators e_i^T, f_i^T are in fact first order differential operators.

Proposition 6.1.2. *We have*

$$\begin{aligned} e_i^T &= \sum_{\theta \in \mathbb{E}_{i-1}} x_\theta \partial_{\theta \star i}, \\ h_i^T &= \sum_{\theta \in \mathbb{E}_i} x_\theta \partial_\theta - \sum_{\theta \in \mathbb{E}_{i-1}} x_\theta \partial_\theta + \sum_{\theta \in \mathbb{B}_i} x_\theta \partial_\theta - \sum_{\theta \in \mathbb{B}_{i+1}} x_\theta \partial_\theta + c_i, \\ f_i^T &= \sum_{\theta \in \mathbb{B}_{i+1}} x_{i \star \theta} \partial_\theta - \sum_{\theta \in \mathbb{E}_{i-1}} x_{\theta \star i} \partial_\theta + x_i \left(\sum_{\theta \in \mathbb{B}_i} x_\theta \partial_\theta - \sum_{\theta \in \mathbb{B}_{i+1}} x_\theta \partial_\theta + c_i \right), \end{aligned} \tag{16}$$

where $\partial_0 = 0$ and $c_i = \langle i', (2-2g)\rho + \alpha_0 \rangle$.

Proof. Direct calculations. \square

It is clear that the Serre relations for the operators e_i, h_i, f_i are equivalent to the Serre relations for the operators e_i^T, h_i^T, f_i^T . Hence the Theorem 3 follows from the following.

Theorem 4. *The operators (16) provide the vector space \mathbf{N} with a structure of $\widehat{\mathfrak{sl}}_n$ -module with the lowest weight c_i .*

The proof of the Theorem 4 will take the rest of the section.

6.2. Check of the relations. Now we can apply the Proposition 6.1.2 to verify the Serre relations. We begin with the following useful notation.

Given a raiz $\theta \in \mathbf{B}_i - \{0\} \subset R^+$ beginning at i we define the differential operators

$$\mathbf{E}(\theta) = \sum_{\vartheta \in \mathbf{E}_{i-1}} x_{\theta * \vartheta} \partial_{\vartheta} \quad \text{and} \quad \tilde{\mathbf{E}}(\theta) = \sum_{\vartheta \in \mathbf{E}_{i-1}} x_{\vartheta} \partial_{\theta * \vartheta}.$$

Similarly, given a raiz $\theta \in \mathbf{E}_i - \{0\} \subset R^+$ ending at i we define

$$\mathbf{B}(\theta) = \sum_{\vartheta \in \mathbf{B}_{i+1}} x_{\theta * \vartheta} \partial_{\vartheta}.$$

Finally we define

$$\mathbf{E}_i = \sum_{\theta \in \mathbf{E}_{i-1}} x_{\theta} \partial_{\theta}, \quad \mathbf{B}_i = \sum_{\theta \in \mathbf{B}_{i+1}} x_{\theta} \partial_{\theta}$$

and

$$\Delta_i = \mathbf{B}_{i-1} + \mathbf{B}_i + c_i.$$

Using this notation we can write our operators in a more compact form

$$\begin{aligned} e_i^T &= \tilde{\mathbf{E}}(i), \\ h_i^T &= \mathbf{E}_{i+1} - \mathbf{E}_i + \Delta_i, \\ f_i^T &= \mathbf{B}(i) - \mathbf{E}(i) + x_i \Delta_i. \end{aligned} \tag{17}$$

We will make also the following agreement:

$$\text{if } \theta \notin R_+ \text{ then } \mathbf{E}(\theta) = \tilde{\mathbf{E}}(\theta) = \mathbf{B}(\theta) = x_{\theta} = 0.$$

Lemma 6.2.1. *We have*

$$\begin{aligned} [\tilde{\mathbf{E}}(\theta_1), \tilde{\mathbf{E}}(\theta_2)] &= \tilde{\mathbf{E}}(\theta_1 * \theta_2) - \tilde{\mathbf{E}}(\theta_2 * \theta_1), \\ [\mathbf{E}(\theta_1), \mathbf{E}(\theta_2)] &= \mathbf{E}(\theta_2 * \theta_1) - \mathbf{E}(\theta_1 * \theta_2), \\ [\mathbf{B}(\theta_1), \mathbf{B}(\theta_2)] &= \mathbf{B}(\theta_1 * \theta_2) - \mathbf{B}(\theta_2 * \theta_1), \\ [\mathbf{B}_i, \mathbf{B}_j] &= 0, \\ [\mathbf{B}(\theta), \mathbf{B}_i] &= \begin{cases} \mathbf{B}(\theta), & \text{if } \theta \in \mathbf{E}_i \setminus \mathbf{B}_{i+1}, \\ -\mathbf{B}(\theta), & \text{if } \theta \in \mathbf{B}_{i+1} \setminus \mathbf{E}_i, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Easy. □

Lemma 6.2.2. *We have*

$$\begin{aligned} [\tilde{\mathbf{E}}(i), \mathbf{B}(j)] &= 0, \\ [\tilde{\mathbf{E}}(i), \mathbf{B}_j] &= 0, \\ [\tilde{\mathbf{E}}(i), \mathbf{E}(j)] &= \delta_{ij}(\mathbf{E}_i - \mathbf{E}_{i+1}), \\ [\tilde{\mathbf{E}}(i), x_j] &= \delta_{ij}. \end{aligned}$$

Proof. Easy. □

Lemma 6.2.3. *We have*

$$\begin{aligned} [\mathbf{E}(\theta_1), \mathbf{B}(\theta_2)] &= 0, \\ [\mathbf{E}(\theta), \mathbf{B}_i] &= 0, \\ [\mathbf{E}(\theta_1), x_{\theta_2}] &= x_{\theta_2 \star \theta_1}, \\ [\mathbf{B}(\theta_1), x_{\theta_2}] &= x_{\theta_1 \star \theta_2}, \\ [\mathbf{B}_i, x_\theta] &= \begin{cases} x_\theta, & \text{if } \theta \in \mathsf{B}_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Easy. □

Lemma 6.2.4. *We have*

$$\begin{aligned} [\Delta_i, \mathbf{E}(\theta)] &= 0; \\ [\Delta_i, \tilde{\mathbf{E}}(\theta)] &= 0; \\ [\Delta_i, \mathbf{B}(\theta)] &= \langle i', \dim \theta \rangle \mathbf{B}(\theta); \\ [\Delta_i, x_\theta] &= \begin{cases} x_\theta, & \text{if } \theta \in \mathsf{B}_i, \\ -x_\theta, & \text{if } \theta \in \mathsf{B}_{i+1}, \\ 0, & \text{otherwise;} \end{cases} \\ [\Delta_i, \Delta_j] &= 0. \end{aligned}$$

Proof. Easy. □

Now we are ready to check the relations.

6.3. Serre relations for e_i^T .

Proposition 6.3.1. *If $j \neq i \pm 1$ then $[e_i^T, e_j^T] = 0$.*

If $j = i \pm 1$ and $n \neq 2$ then $\mathbf{ad}(e_i^T)^2 e_j^T = 0$.

If $j = i + 1$ and $n = 2$ then $\mathbf{ad}(e_i^T)^3 e_j^T = 0$.

Proof. In the first case $i \star j \notin R^+$ and $j \star i \notin R^+$ hence according to (17) and to the Lemma 6.2.1 we get

$$[\tilde{\mathbf{E}}(i), \tilde{\mathbf{E}}(j)] = 0.$$

If $j = i + 1$ and $n \neq 2$ then

$$[\tilde{\mathbf{E}}(i), [\tilde{\mathbf{E}}(i), \tilde{\mathbf{E}}(j)]] = [\tilde{\mathbf{E}}(i), \tilde{\mathbf{E}}(i \star j)] = 0$$

and if $j = i - 1$ and $n \neq 2$ then

$$[\tilde{\mathbf{E}}(i), [\tilde{\mathbf{E}}(i), \tilde{\mathbf{E}}(j)]] = -[\tilde{\mathbf{E}}(i), \tilde{\mathbf{E}}(j \star i)] = 0.$$

Finally, if $j = i + 1$ and $n = 2$ then

$$\begin{aligned} [\tilde{\mathbf{E}}(i), [\tilde{\mathbf{E}}(i), [\tilde{\mathbf{E}}(i), \tilde{\mathbf{E}}(j)]]] &= [\tilde{\mathbf{E}}(i), [\tilde{\mathbf{E}}(i), \tilde{\mathbf{E}}(i \star j) - \tilde{\mathbf{E}}(j \star i)]] = \\ &= -2[\tilde{\mathbf{E}}(i), \tilde{\mathbf{E}}(i \star j \star i)] = 0. \end{aligned}$$

□

6.4. Commutators of e_i^T and f_j^T .

Proposition 6.4.1. *We have $[e_i^T, f_j^T] = \delta_{ij} h_i^T$.*

Proof. According to (17) and to the Lemma 6.2.2 we have

$$[\tilde{\mathbf{E}}(i), \mathbf{B}(j) - \mathbf{E}(j) + x_j \Delta_j] = -\delta_{ij}(\mathbf{E}_i - \mathbf{E}_{i+1}) + \delta_{ij} \Delta_j = \delta_{ij} h_i^T.$$

□

6.5. Serre relations for f_i^T .

Proposition 6.5.1. *If $j \neq i \pm 1$ then $[f_i^T, f_j^T] = 0$.*

Proof. According to (17) and to the Lemmas 6.2.1, 6.2.3 and 6.2.4 we have

$$[f_i^T, f_j^T] = [\mathbf{B}(i) - \mathbf{E}(i) + x_i \Delta_i, \mathbf{B}(j) - \mathbf{E}(j) + x_j \Delta_j] = 0.$$

□

Proposition 6.5.2. *If $j = i \pm 1$ and $n \neq 2$ then $\mathbf{ad}(f_i^T)^2 f_j^T = 0$.*

Proof. Assume that $j = i + 1$. We denote

$$D = [f_i^T, f_j^T] = [\mathbf{B}(i) - \mathbf{E}(i) + x_i \Delta_i, \mathbf{B}(j) - \mathbf{E}(j) + x_j \Delta_j].$$

Then according to (17) and to the Lemmas 6.2.1, 6.2.3 and 6.2.4 we have

$$D = \mathbf{B}(i \star j) - \mathbf{E}(i \star j) + x_j \mathbf{B}(i) - x_i \mathbf{B}(j) + x_{i \star j} \Delta_i + (x_{i \star j} - x_i x_j) \Delta_j.$$

It suffices to show that $[f_i^T, D] = [f_j^T, D] = 0$. Applying once more (17) and the Lemmas 6.2.1, 6.2.3 and 6.2.4 we get

$$\begin{aligned} [\mathbf{B}(i), D] &= x_{i \star j} \mathbf{B}(i) - x_i \mathbf{B}(i \star j) - 2x_{i \star j} \mathbf{B}(i) + (x_{i \star j} - x_i x_j) \mathbf{B}(i) - x_i x_{i \star j} \Delta_j; \\ [\mathbf{E}(i), D] &= 0; \\ [x_i \Delta_i, D] &= x_i \mathbf{B}(i \star j) + x_i x_j \mathbf{B}(i) + x_i x_{i \star j} \Delta_j. \end{aligned}$$

Hence $[f_i^T, D] = 0$. Similarly, we get

$$\begin{aligned} [\mathbf{B}(j), D] &= -x_j \mathbf{B}(i \star j) + x_{i \star j} \mathbf{B}(j) - 2(x_{i \star j} - x_i x_j) \mathbf{B}(j); \\ [\mathbf{E}(j), D] &= -x_{i \star j} \mathbf{B}(j) - x_{i \star j} x_j \Delta_j; \\ [x_j \Delta_j, D] &= x_j \mathbf{B}(i \star j) - x_j x_{i \star j} \Delta_j - 2x_i x_j \mathbf{B}(j) + x_j x_{i \star j} \Delta_j - x_j x_{i \star j} \Delta_j. \end{aligned}$$

Hence $[f_j^T, D] = 0$.

□

Proposition 6.5.3. *If $j = i + 1$ and $n = 2$ then $\mathbf{ad}(f_i^T)^3 f_j^T = 0$.*

Proof. We denote

$$D = [f_i^T, f_j^T] = [\mathbf{B}(i) - \mathbf{E}(i) + x_i \Delta_i, \mathbf{B}(j) - \mathbf{E}(j) + x_j \Delta_j].$$

Then according to (17) and to the Lemmas 6.2.1, 6.2.3 and 6.2.4 we have

$$\begin{aligned} D &= \mathbf{B}(i \star j) - \mathbf{B}(j \star i) - \mathbf{E}(i \star j) + \mathbf{E}(j \star i) + \\ &\quad + 2x_j \mathbf{B}(i) - 2x_i \mathbf{B}(j) + (x_{i \star j} - x_{j \star i} + x_i x_j) \Delta_i + (x_{i \star j} - x_{j \star i} - x_i x_j) \Delta_j. \end{aligned}$$

Appling once more (17) and the Lemmas 6.2.1, 6.2.3 and 6.2.4 we get

$$\begin{aligned}
 [\mathbf{B}(i), D] &= -2\mathbf{B}(i \star j \star i) + 2x_{i \star j} \mathbf{B}(i) - 2x_i \mathbf{B}(i \star j) + 2x_i \mathbf{B}(j \star i) + \\
 &\quad + (x_i x_{i \star j} - x_{i \star j \star i}) \Delta_i - 2(x_{i \star j} - x_{j \star i} + x_i x_j) \mathbf{B}(i) - \\
 &\quad - (x_i x_{i \star j} + x_{i \star j \star i}) \Delta_j + 2(x_{i \star j} - x_{j \star i} - x_i x_j) \mathbf{B}(i); \\
 [\mathbf{E}(i), D] &= -2\mathbf{E}(i \star j \star i) + 2x_{j \star i} \mathbf{B}(i) + \\
 &\quad + (x_{i \star j \star i} + x_i x_{j \star i}) \Delta_i + (x_{i \star j \star i} - x_i x_{j \star i}) \Delta_j; \\
 [x_i \Delta_i, D] &= -2x_{i \star j \star i} \Delta_i + 2x_i x_j \mathbf{B}(i) + 2x_i^2 \mathbf{B}(j) + 2x_i x_{j \star i} \Delta_i + \\
 &\quad + x_i (2x_{i \star j} + x_i x_j) \Delta_i + x_i (x_{i \star j} + x_{j \star i}) \Delta_j - \\
 &\quad - x_i (x_{i \star j} - x_{j \star i} - x_i x_j) \Delta_i.
 \end{aligned}$$

Hence $[f_i^T, D] = 2\tilde{D}$, where

$$\begin{aligned}
 \tilde{D} &= -\mathbf{B}(i \star j \star i) + \mathbf{E}(i \star j \star i) + x_i \mathbf{B}(j \star i) - x_i \mathbf{B}(i \star j) + (x_{i \star j} - x_{j \star i} - x_i x_j) \mathbf{B}(i) + \\
 &\quad + x_i^2 \mathbf{B}(j) + (-2x_{i \star j \star i} + x_i x_{i \star j} + x_i x_{j \star i} - x_i^2 x_j) \Delta_i + (-x_{i \star j \star i} + x_i x_{j \star i}) \Delta_j.
 \end{aligned}$$

Appling once more (17) and the Lemmas 6.2.1, 6.2.3 and 6.2.4 we get

$$\begin{aligned}
 [\mathbf{B}(i), \tilde{D}] &= 2x_i \mathbf{B}(i \star j \star i) - (x_{i \star j \star i} + x_i x_{i \star j}) \mathbf{B}(i) + \\
 &\quad + x_i^2 (\mathbf{B}(i \star j) - \mathbf{B}(j \star i)) + (x_i x_{i \star j \star i} - x_i^2 x_{i \star j}) \Delta_i + \\
 &\quad + 2(2x_{i \star j \star i} - x_i x_{i \star j} - x_i x_{j \star i} + x_i^2 x_j) \mathbf{B}(i) + \\
 &\quad + x_i x_{i \star j \star i} \Delta_j + 2(-x_{i \star j \star i} + x_i x_{j \star i}) \mathbf{B}(i); \\
 [\mathbf{E}(i), \tilde{D}] &= (x_{i \star j \star i} - x_i x_{j \star i}) \mathbf{B}(i) + (x_i x_{i \star j \star i} - x_i^2 x_{j \star i}) \Delta_i; \\
 [x_i \Delta_i, \tilde{D}] &= -2x_i \mathbf{B}(i \star j \star i) + x_i^2 \mathbf{B}(j \star i) - x_i^2 \mathbf{B}(i \star j) + x_i x_{i \star j \star i} \Delta_i + \\
 &\quad + x_i (x_{i \star j} + x_{j \star i}) \mathbf{B}(i) - 2x_i (-x_{i \star j} + x_{j \star i} + x_i x_j) \mathbf{B}(i) - \\
 &\quad - x_i^2 x_{j \star i} \Delta_i + x_i (-2x_{i \star j \star i} + 2x_i x_{i \star j} - x_i^2 x_j) \Delta_i + \\
 &\quad + x_i (2x_{i \star j \star i} - x_i x_{i \star j} - x_i x_{j \star i} + x_i^2 x_j) \Delta_i - \\
 &\quad - x_i x_{i \star j \star i} \Delta_j + x_i (-x_{i \star j \star i} + x_i x_{j \star i}) \Delta_i.
 \end{aligned}$$

Hence $[f_i^T, \tilde{D}] = 0$ and the Proposition follows. \square

7. EXTENDING THE ACTION OF $\widehat{\mathfrak{sl}}_n$ ON \mathbf{M} TO $\widehat{\mathfrak{gl}}_n$

In this section we will deal with the Lie algebras $\widehat{\mathfrak{sl}}_n$ for various n . So in order to avoid a confusion we will denote the corresponding set of simple raiz by $I(n)$, the system of raiz by $R^+(n)$, and the raiz lattice by $Y(n)$.

7.1. $\widehat{\mathfrak{sl}}_n$ and $\widehat{\mathfrak{sl}}_{kn}$. Recall that the group $\mathbb{Z}/n\mathbb{Z}$ acts on the Lie algebra $\widehat{\mathfrak{sl}}_n$ by outer automorphisms. This group acts also on the set of simple raiz $I(n)$, on the raiz system $R^+(n)$, and on the raiz lattice $Y(n)$. We denote the action of an element $a \in \mathbb{Z}/n\mathbb{Z}$ by τ_a .

Lemma 7.1.1. *For all integers $k \geq 2$, $n \geq 2$ there is a Lie algebras homomorphism $\mu : \widehat{\mathfrak{sl}}_n \rightarrow \widehat{\mathfrak{sl}}_{kn}$ defined on the Chevalley generators as follows*

$$\mu(e_i) = \sum_{a \in n\mathbb{Z}/(kn)\mathbb{Z}} e_{\tau_a(i)}, \quad \mu(f_i) = \sum_{a \in n\mathbb{Z}/(kn)\mathbb{Z}} f_{\tau_a(i)}, \quad \mu(h_i) = \sum_{a \in n\mathbb{Z}/(kn)\mathbb{Z}} h_{\tau_a(i)}.$$

Proof. Evident. \square

On the other hand, the identifications

$$R^+(kn) = (\mathbb{Z} \times_{\leq} \mathbb{Z})/(kn)\mathbb{Z}, \quad R^+(n) = (\mathbb{Z} \times_{\leq} \mathbb{Z})/n\mathbb{Z}$$

give rise to the projection

$$\zeta : R^+(kn) \rightarrow R^+(n), \quad (p, q) \bmod kn \mapsto (p, q) \bmod n.$$

The projection ζ in its turn induces a morphism of algebras

$$\zeta : \mathbf{N}(kn) \rightarrow \mathbf{N}(n), \quad x_{\vartheta} \mapsto x_{\zeta(\vartheta)}.$$

The following Proposition will be very important below.

Proposition 7.1.2. *The homomorphism ζ is a homomorphism of $\widehat{\mathfrak{sl}}_n$ -modules, that is for all $\xi \in \widehat{\mathfrak{sl}}_n$, $P(x) \in \mathbf{N}(kn)$ we have*

$$\xi \cdot \zeta(P) = \zeta(\mu(\xi) \cdot P). \quad (18)$$

where \cdot stands for the action of the Lie algebra $\widehat{\mathfrak{sl}}_n$ (resp. $\widehat{\mathfrak{sl}}_{kn}$) on the space $\mathbf{N}(kn)$ (resp. $\mathbf{N}(n)$).

Proof. Since μ is a Lie algebra homomorphism it suffices to check (18) only for the Chevalley generators e_i and f_i . On the other hand, both e_i and f_i act as first order differential operators, hence both LHS and RHS of (18) are first order differential operators on $\mathbf{N}(kn)$ with values in $\mathbf{N}(n)$. Hence it suffices to check (18) only for $P = x_{\vartheta}$, $\vartheta \in R^+(kn) \cup \{0\}$, and this can be done straightforwardly. \square

In fact, the homomorphism $\mu : \widehat{\mathfrak{sl}}_n \rightarrow \widehat{\mathfrak{sl}}_{kn} \subset \mathbf{Diff}(\mathbf{N}(kn))$ can be extended to some bigger subalgebra of $\mathbf{Diff}(\mathbf{N}(n))$.

Definition 7.1.3. We define

$$\mu(\tilde{\mathbf{E}}(\theta)) = \sum_{\vartheta \in \zeta^{-1}(\theta)} \tilde{\mathbf{E}}(\vartheta), \quad \theta \in R^+(n).$$

The analog of the property (18) is satisfied for the operators $\xi = \tilde{\mathbf{E}}(\theta)$:

Proposition 7.1.4. *We have*

$$\tilde{\mathbf{E}}(\theta) \cdot \zeta(P) = \zeta(\mu(\tilde{\mathbf{E}}(\theta)) \cdot P). \quad (19)$$

Proof. Again it suffices to check (19) only for $P = x_{\vartheta}$, $\vartheta \in R^+(kn) \cup \{0\}$ and this can be done straightforwardly. \square

7.2. The polynomial P_n . Let $\alpha_n = \sum_{i \in I(n)} i \in Y(n)$ be the element of the lattice $Y(n)$, corresponding to the sequence $(\dots, 1, 1, 1, 1, \dots)$ of integers.

Consider the following element of the space $\mathbf{N}(n)$

$$P_n = \sum_{\kappa = \{\{\theta_1, \dots, \theta_m\}\} \in \mathfrak{K}(\alpha_n)} P_n^{\kappa} x^{\kappa} = \sum_{\kappa = \{\{\theta_1, \dots, \theta_m\}\} \in \mathfrak{K}(\alpha_n)} (-1)^{\mathbf{K}(\kappa)+1} x_{\theta_1} \cdot \dots \cdot x_{\theta_m}.$$

This element of the space $\mathbf{N}(n)$ has the following properties.

Lemma 7.2.1. *Let $\theta \in R^+(n)$. We have*

$$\tilde{\mathbf{E}}(\theta) \cdot P_n = \begin{cases} 1, & \text{if } \dim \theta = \alpha_n \\ 0, & \text{otherwise} \end{cases}$$

Proof. The first case is evident, so assume that $\dim \theta \neq \alpha_n$. It is clear that

$$\tilde{\mathbf{E}}(\theta) \cdot P_n = \sum_{\kappa \in \mathfrak{K}(\alpha_n - \dim \theta)} \lambda_\kappa x^\kappa$$

with some coefficients $\lambda_\kappa \in \mathbb{Z}$. Hence we can assume also that $\alpha_n - \dim \theta \in \mathbb{N}[I]$.

It suffices to show that all coefficients λ_κ vanish. Fix some $\kappa = \{\{\theta_1, \dots, \theta_m\}\}$ and assume that $\theta \in \mathbf{B}_i$. It is clear that the partition κ has exactly one element contained in \mathbf{E}_{i-1} . Assume that it is θ_m .

Let $\kappa' = \{\{\theta_1, \dots, \theta_m \star \theta\}\}$ and $\kappa'' = \{\{\theta_1, \dots, \theta_m, \theta\}\}$. It is clear that

$$\lambda_\kappa = P_n^{\kappa'} + P_n^{\kappa''},$$

but $\mathsf{K}(\kappa'') = \mathsf{K}(\kappa') + 1$, hence $\lambda_\kappa = 0$. \square

Let $f'_i = f_i^T - c_i x_i = \mathbf{B}(i) - \mathbf{E}(i) + x_i(\mathbf{B}_{i-1} - \mathbf{B}_i)$.

Lemma 7.2.2. *We have*

$$f'_i \cdot P_n = 0$$

for all $i \in I(n)$.

Proof. It is clear that

$$f'_i \cdot P_n = \sum_{\kappa \in \mathfrak{K}(\alpha_n + i)} \lambda_\kappa x^\kappa$$

with some coefficients $\lambda_\kappa \in \mathbb{Z}$. We will check that all coefficients λ_κ vanish. We have the following cases to consider:

- 1) $\kappa = \{\{i, i\}\} \cup \kappa'$;
- 2) $\kappa = \{\{i, i \star \theta\}\} \cup \kappa'$, $\theta \in \mathbf{B}_{i+1}$;
- 3) $\kappa = \{\{i, \theta \star i\}\} \cup \kappa'$, $\theta \in \mathbf{E}_{i-1}$;
- 4) $\kappa = \{\{i, \theta\}\} \cup \kappa'$, $\theta \geq i$ but $\theta \notin \mathbf{B}_i \cup \mathbf{E}_i$;
- 5) $\kappa = \{\{i \star \theta_1, \theta_2 \star i\}\} \cup \kappa'$, $\theta_1 \in \mathbf{B}_{i+1}$, $\theta_2 \in \mathbf{E}_{i-1}$.

In the first case we have

$$\lambda_\kappa = P_n^{\{\{i\}\} \cup \kappa'}((\mathbf{B}_{i-1} - \mathbf{B}_i)(x^{\{\{i\}\} \cup \kappa'})) = 0.$$

In the second case we have

$$\begin{aligned} \lambda_\kappa &= P_n^{\{\{i, \theta\}\} \cup \kappa'} + P_n^{\{\{i \star \theta\}\} \cup \kappa'}((\mathbf{B}_{i-1} - \mathbf{B}_i)(x^{\{\{i \star \theta\}\} \cup \kappa'})) = \\ &= P_n^{\{\{i, \theta\}\} \cup \kappa'} + P_n^{\{\{i \star \theta\}\} \cup \kappa'} \end{aligned}$$

which is zero because $\mathsf{K}(\{\{i, \theta\}\} \cup \kappa') = \mathsf{K}(\{\{i \star \theta\}\} \cup \kappa') + 1$.

In the third case we have

$$\begin{aligned} \lambda_\kappa &= -P_n^{\{\{i, \theta\}\} \cup \kappa'} + P_n^{\{\{\theta \star i\}\} \cup \kappa'}((\mathbf{B}_{i-1} - \mathbf{B}_i)(x^{\{\{\theta \star i\}\} \cup \kappa'})) = \\ &= -P_n^{\{\{i, \theta\}\} \cup \kappa'} - P_n^{\{\{i \star \theta\}\} \cup \kappa'} \end{aligned}$$

which is zero because $\mathsf{K}(\{\{i, \theta\}\} \cup \kappa') = \mathsf{K}(\{\{i \star \theta\}\} \cup \kappa') + 1$.

In the fourth case we have

$$\lambda_\kappa = P_n^{\{\{\theta\}\} \cup \kappa'}((\mathbf{B}_{i-1} - \mathbf{B}_i)(x^{\{\{\theta\}\} \cup \kappa'})) = 0.$$

Finally, in the fifth case we have

$$\lambda_\kappa = P_n^{\{\{\theta_1, \theta_2 \star i\}\} \cup \kappa'} - P_n^{\{\{i \star \theta_1, \theta_2\}\} \cup \kappa'},$$

which is zero because $\mathsf{K}(\{\{\theta_1, \theta_2 \star i\}\} \cup \kappa') = \mathsf{K}(\{\{i \star \theta_1, \theta_2\}\} \cup \kappa')$. \square

7.3. $\widehat{\mathfrak{gl}}_n$ -structure. Recall that

$$\widehat{\mathfrak{gl}}_n/\mathbf{c} = \widehat{\mathfrak{sl}}_n/\mathbf{c} \oplus \mathbf{Heis}/\mathbf{c},$$

where \mathbf{c} is the central element and \mathbf{Heis} is the Heisenberg algebra, that is

$$\mathbf{Heis} = \mathbb{Q}(\dots, \mathbf{a}_{-2}, \mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \dots) \oplus \mathbb{Q}\mathbf{c}$$

with

$$[\mathbf{a}_p, \mathbf{a}_q] = \delta_{p,-q} p \mathbf{c}, \quad [\mathbf{a}_p, \mathbf{c}] = 0.$$

Thus, if we want to extend an action of $\widehat{\mathfrak{sl}}_n$ with the central charge c_0 to the action of $\widehat{\mathfrak{gl}}_n$, we have to construct an action of the Heisenberg algebra with the central charge nc_0 , commuting with the action of $\widehat{\mathfrak{sl}}_n$.

Definition 7.3.1. (compare [11]) We define

$$\mathbf{a}_p = \sum_{\dim \theta = p\alpha_n} \tilde{\mathbf{E}}(\theta), \quad \mathbf{a}_{-p} = c_0 \zeta(P_{pn}), \quad p > 0, \quad \text{and} \quad \mathbf{a}_0 = c_0 \mathbf{id}, \quad \mathbf{c} = nc_0 \mathbf{id}. \quad (20)$$

Proposition 7.3.2. *The operators \mathbf{a}_p and \mathbf{c} satisfy the relations of the Heisenberg algebra.*

Proof. The equality $[\mathbf{a}_p, \mathbf{a}_q] = 0$ for $p, q < 0$ is evident and for $p, q > 0$ it follows from the Lemma 6.2.1. Hence it suffices to consider the case $p > 0, q < 0$. In this case we have

$$\begin{aligned} [\mathbf{a}_p, \mathbf{a}_q] &= c_0 \sum_{\dim \theta = p\alpha_n} \tilde{\mathbf{E}}(\theta) \cdot \zeta(P_{-qn}) = c_0 \sum_{\dim \theta = p\alpha_n} \zeta(\mu(\tilde{\mathbf{E}}(\theta)) \cdot P_{-qn}) = \\ &= c_0 \sum_{\dim \zeta(\vartheta) = p\alpha_n} \zeta(\tilde{\mathbf{E}}(\vartheta) \cdot P_{-qn}) = \begin{cases} pnc_0, & \text{if } p = -q \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

because $\dim \zeta(\vartheta) = p\alpha_n$ is equivalent to $\dim \vartheta = \alpha_{pn}$. \square

Thus it remains to check that \mathbf{a}_p commute with the action of $\widehat{\mathfrak{sl}}_n$. We will need the following Lemma.

Lemma 7.3.3. *We have*

$$\begin{aligned} \sum_{\dim \theta = p\alpha_n} [\tilde{\mathbf{E}}(\theta), \mathbf{B}(i)] &= \partial_{\theta_1} + x_i \partial_{\theta_2}, \\ \sum_{\dim \theta = p\alpha_n} [\tilde{\mathbf{E}}(\theta), \mathbf{E}(i)] &= \partial_{\theta_1} + x_i \partial_{\theta_3}, \\ \sum_{\dim \theta = p\alpha_n} [\tilde{\mathbf{E}}(\theta), x_i \Delta_i] &= x_i (\partial_{\theta_3} - \partial_{\theta_2}), \end{aligned}$$

where $\theta_1, \theta_2, \theta_3 \in R^+(n)$ are defined by the following properties

$$\dim \theta_1 = p\alpha_n - i, \quad \dim \theta_2 = p\alpha_n \text{ and } \theta_2 \in \mathbf{B}_{i+1}, \quad \dim \theta_3 = p\alpha_n \text{ and } \theta_3 \in \mathbf{B}_i.$$

Proof. Evident. \square

Proposition 7.3.4. *The operator \mathbf{a}_p commutes with $\widehat{\mathfrak{sl}}_n$ for all $p > 0$.*

Proof. It suffices to check that the operators \mathbf{a}_p commute with Chevalley generators e_i^T and f_i^T . It follows from the Lemma 6.2.1 that

$$[\mathbf{a}_p, e_i^T] = \sum_{\dim \theta = p\alpha_n} [\tilde{\mathbf{E}}(\theta), \tilde{\mathbf{E}}(i)] = 0$$

and the Lemma 7.3.3 implies that $[\mathbf{a}_p, f_i^T] = 0$. \square

Proposition 7.3.5. *The operator \mathbf{a}_{-p} commutes with $\widehat{\mathfrak{sl}}_n$ for all $p > 0$.*

Proof. We have

$$\begin{aligned} [e_i^T, \mathbf{a}_{-p}] &= c_0 e_i^T \cdot \zeta(P_{pn}) = c_0 \zeta(\mu(e_i^T) \cdot P_{pn}) = \\ &= c_0 \sum \zeta(e_{\tau_a(i)}^T \cdot P_{pn}) = c_0 \sum \zeta(\tilde{\mathbf{E}}(\tau_a(i)) \cdot P_{pn}) = 0. \end{aligned}$$

On the other hand

$$[f_i^T, \mathbf{a}_{-p}] = c_0 f_i \cdot \zeta(P_{pn}) - c_0 c_i x_i \zeta(P_{pn})$$

and

$$\begin{aligned} f_i^T \cdot \zeta(P_{pn}) &= \zeta(\mu(f_i^T) \cdot P_{pn}) = \sum \zeta(f_{\tau_a(i)}^T \cdot P_{pn}) = \\ &= \sum \zeta((f'_{\tau_a(i)} + c_i x_{\tau_a(i)}) \cdot P_{pn}) = \sum c_i \zeta(x_{\tau_a(i)}) \zeta(P_{pn}) = c_i x_i \zeta(P_{pn}) \end{aligned}$$

and the Proposition follows. \square

Thus we have proved the following.

Theorem 5. *The operators (20) extend the structure of $\widehat{\mathfrak{sl}}_n$ -module of the vector space \mathbf{N} to that of $\widehat{\mathfrak{gl}}_n$ -module.*

Corollary 7.3.6. *The vector space \mathbf{M} has the natural structure of a $\widehat{\mathfrak{gl}}_n$ -module.*

Remark 7.3.7. It is easy to show that the action of the operators \mathbf{a}_p with $p > 0$ on the vector space \mathbf{M} can be described geometrically. Namely, the operator \mathbf{a}_p is given by the correspondence

$$\mathfrak{E}_\alpha^{p\alpha_n} = \{(E_\bullet, E'_\bullet) \in K_\alpha \times K_{\alpha+p\alpha_n} \mid E'_\bullet \subset E_\bullet \text{ and } \mathbf{supp}(E'_\bullet/E_\bullet) = \{x\} \in C\}$$

in the same way as the operator e_i is given by the correspondence \mathfrak{E}_α^i . It is tempting also to conjecture that the operator \mathbf{a}_{-p} is given by the same correspondence $\mathfrak{E}_\alpha^{p\alpha_n}$ in the same way as the operator f_i is given by the correspondence \mathfrak{E}_α^i . However, it is rather difficult to check, because the correspondence $\mathfrak{E}_\alpha^{p\alpha_n}$ has a lot of components with dimensions in the range $|\alpha|, \dots, |\alpha + p\alpha_n|$ and all of them should contribute to this operator.

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INDEPENDENT MOSCOW UNIVERSITY, BOLSHOJ VLASJEVSKIJ PEREULOK, DOM 11, MOSCOW 121002 RUSSIA

E-mail address: `fnklberg@mccme.ru`

INDEPENDENT MOSCOW UNIVERSITY, BOLSHOJ VLASJEVSKIJ PEREULOK, DOM 11, MOSCOW 121002 RUSSIA

E-mail address: `sasha@kuznetsov.mccme.ru`